CS38
Introduction to Algorithms

Lecture 15

May 20, 2014

Outline

- Linear programming
  - simplex algorithm
  - LP duality
  - ellipsoid algorithm

* slides from Kevin Wayne

Linear programming

Standard Form LP

"Standard form" LP.
- Input: real numbers a_{ij}, c_j, b_i.
- Output: real numbers x_j.
- n = # decision variables, m = # constraints.
- Maximize linear objective function subject to linear inequalities.

$$
\begin{array}{llrcll}
\text{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\
 & \text{s. t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\
 & & x_j & \ge & 0 & 1 \le j \le n
\end{array}
\qquad
\begin{array}{llrcl}
\text{(P)} & \max & c^T x & & \\
 & \text{s. t.} & Ax & = & b \\
 & & x & \ge & 0
\end{array}
$$

Linear. No x^2, xy, $\arccos(x)$, etc.
Programming. Planning (term predates computer programming).

Brewery Problem: Converting to Standard Form

Original input.

$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s. t.} & 5A & + & 15B & \le & 480 \\
 & 4A & + & 4B & \le & 160 \\
 & 35A & + & 20B & \le & 1190 \\
 & A & , & B & \ge & 0
\end{array}
$$

Standard form.
- Add slack variable for each inequality.
- Now a 5-dimensional problem.

$$
\begin{array}{lrcrcccccr}
\max & 13A & + & 23B & & & & & & \\
\text{s. t.} & 5A & + & 15B & + & S_C & & & = & 480 \\
 & 4A & + & 4B & & & + S_H & & = & 160 \\
 & 35A & + & 20B & & & & + S_M & = & 1190 \\
 & A & , & B & , & S_C & , S_H & , S_M & \ge & 0
\end{array}
$$

Equivalent Forms

Easy to convert variants to standard form.

$$
\begin{array}{llrcl}
\text{(P)} & \max & c^T x & & \\
 & \text{s. t.} & Ax & = & b \\
 & & x & \ge & 0
\end{array}
$$

Less than to equality:

$x + 2y - 3z \le 17 \quad \Rightarrow \quad x + 2y - 3z + s = 17, s \ge 0$

Greater than to equality:

$x + 2y - 3z \ge 17 \quad \Rightarrow \quad x + 2y - 3z - s = 17, s \ge 0$

Min to max:

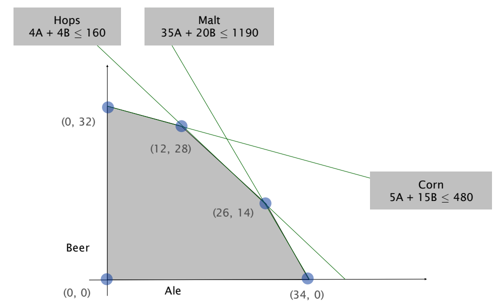
$\min\ x + 2y - 3z \quad \Rightarrow \quad \max\ -x - 2y + 3z$

Unrestricted to nonnegative:

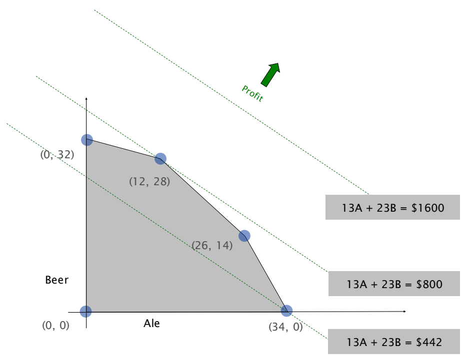
x unrestricted $\quad \Rightarrow \quad x = x^+ - x^-,\ x^+ \ge 0, x^- \ge 0$

Linear programming geometric perspective

Brewery Problem: Feasible Region

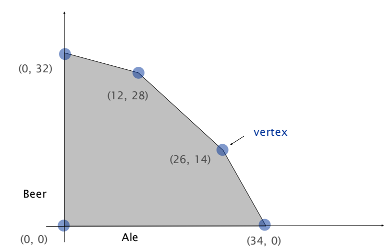


Brewery Problem: Objective Function



Brewery Problem: Geometry

Brewery problem observation. Regardless of objective function coefficients, an optimal solution occurs at a vertex.

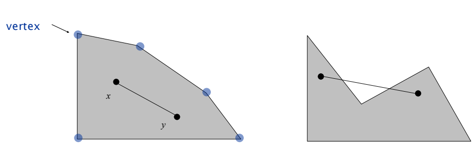


Convexity

Convex set. If two points x and y are in the set, then so is $\lambda x + (1-\lambda) y$ for $0 \le \lambda \le 1$.

convex combination

Vertex. A point x in the set that can't be written as a strict convex combination of two distinct points in the set.



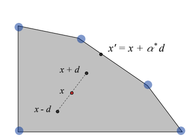
Observation. LP feasible region is a convex set.

Geometric perspective

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

$$(P) \quad \max \; c^T x \quad \text{s.t.} \quad Ax = b, \quad x \ge 0$$

Intuition. If x is not a vertex, move in a non-decreasing direction until you reach a boundary. Repeat.



Geometric perspective

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

Pf.
- Suppose x is an optimal solution that is not a vertex.
- There exist direction d νοτ εθυαλ το 0 such that $x \pm d \in P$.
- $A\,d = 0$ because $A(x \pm d) = b$.
- Assume $c^T d \geq 0$ (by taking either d or $-d$).
- Consider $x + \lambda d$, $\lambda > 0$:

Case 1. [there exists j such that $d_j < 0$]
- Increase λ to λ^* until first new component of $x + \lambda d$ hits 0.
- $x + \lambda^* d$ is feasible since $A(x + \lambda^* d) = Ax = b$ and $x + \lambda^* y \geq 0$.
- $x + \lambda^* d$ has one more zero component than x.
- $c^T x' = c^T (x + \lambda^* d) = c^T x + \lambda^* c^T d \geq c^T x$.

$d_k = 0$ whenever $x_k = 0$ because $x \pm d \in P$

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Geometric perspective

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

Pf.
- Suppose x is an optimal solution that is not a vertex.
- There exist direction d νοτ εθυαλ το 0 such that $x \pm d \in P$.
- $A\,d = 0$ because $A(x \pm d) = b$.
- Assume $c^T d \geq 0$ (by taking either d or $-d$).
- Consider $x + \lambda d$, $\lambda > 0$:

Case 2. [$d_j \geq 0$ for all j]
- $x + \lambda\, d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda\, d) = b$ and $x + \lambda\, d \geq x \geq 0$.
- As $\lambda \to \infty$, $c^T(x + \lambda\, d) \to \infty$ because $c^T d > 0$.

if $c^T d = 0$, choose d so that case 1 applies

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Linear programming
linear algebraic perspective

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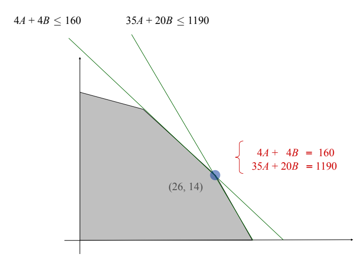
Intuition

Intuition. A vertex in R^m is uniquely specified by m linearly independent equations.



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Basic Feasible Solution

Theorem. Let $P = \{ x : Ax = b, x \geq 0 \}$. For $x \in P$, define $B = \{ j : x_j > 0 \}$. Then x is a vertex iff A_B has linearly independent columns.

Notation. Let B = set of column indices. Define A_B to be the subset of columns of A indexed by B.

Ex.

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 16 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \{1, 3\}, \quad A_B = \begin{bmatrix} 2 & 3 \\ 7 & 2 \\ 0 & 0 \end{bmatrix}$$

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Basic Feasible Solution

Theorem. Let $P = \{ x : Ax = b, x \geq 0 \}$. For $x \in P$, define $B = \{ j : x_j > 0 \}$. Then x is a vertex iff A_B has linearly independent columns.

Pf. $\Leftarrow$
- Assume x is not a vertex.
- There exist direction d not equal to 0 such that $x \pm d \in P$.
- $A\,d = 0$ because $A(x \pm d) = b$.
- Define $B' = \{ j : d_j$ νοτ εθυαλ το $0 \}$.
- $A_{B'}$ has linearly dependent columns since d not equal to 0.
- Moreover, $d_j = 0$ whenever $x_j = 0$ because $x \pm d \geq 0$.
- Thus $B' \subseteq B$, so $A_{B'}$ is a submatrix of A_B.
- Therefore, A_B has linearly dependent columns.

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Basic Feasible Solution

Theorem. Let $P = \{ x : Ax = b, x \geq 0 \}$. For $x \in P$, define $B = \{ j : x_j > 0 \}$. Then x is a vertex iff A_B has linearly independent columns.

Pf. $\Rightarrow$

- Assume A_B has linearly dependent columns.
- There exist d not equal to 0 such that $A_B\, d = 0$.
- Extend d to R^n by adding 0 components.
- Now, $A\, d = 0$ and $d_j = 0$ whenever $x_j = 0$.
- For sufficiently small λ, $x \pm \lambda d \in P \Rightarrow x$ is not a vertex.

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Basic Feasible Solution

Theorem. Given $P = \{ x : Ax = b, x \geq 0 \}$, x is a vertex iff there exists $B \subseteq \{ 1, \ldots, n \}$ such that $|B| = m$ and:

- A_B is nonsingular.
- $x_B = A_B^{-1}\, b \geq 0$. (basic feasible solution)
- $x_N = 0$.

Pf. Augment A_B with linearly independent columns (if needed). ▪

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 16 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \{ 1, 3, 4 \}, \quad A_B = \begin{bmatrix} 2 & 3 & 0 \\ 7 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Assumption. $A \in R^{m \times n}$ has full row rank.

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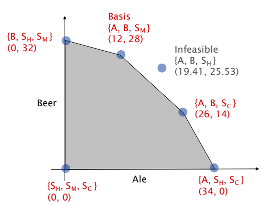
Basic Feasible Solution: Example

Basic feasible solutions.

$$\begin{array}{lrcrcccccl} \max & 13A & + & 23B & & & & & & \\ \text{s. t.} & 5A & + & 15B & + & S_C & & & = & 480 \\ & 4A & + & 4B & & & + & S_H & = & 160 \\ & 35A & + & 20B & & & + & S_M & = & 1190 \\ & A & , & B & , & S_C & , & S_H\ ,\ S_M & \geq & 0 \end{array}$$



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Simplex algorithm

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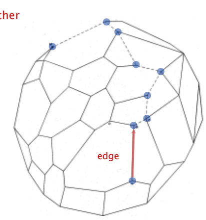
Simplex Algorithm: Intuition

Simplex algorithm. [George Dantzig 1947] Move from BFS to adjacent BFS, without decreasing objective function.

replace one basic variable with another



Greedy property. BFS optimal iff no adjacent BFS is better.

Challenge. Number of BFS can be exponential!

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Simplex Algorithm: Initialization

$$\begin{array}{rcrcccccccl} \multicolumn{11}{l}{\max Z \text{ subject to}} \\ 13A & + & 23B & & & & & & - & Z = & 0 \\ 5A & + & 15B & + & S_C & & & & & = & 480 \\ 4A & + & 4B & & & + & S_H & & & = & 160 \\ 35A & + & 20B & & & & & + & S_M & = & 1190 \\ A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0 \end{array}$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

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Simplex Algorithm: Pivot 1

$$\begin{array}{l} \max\ Z \text{ subject to} \\ \begin{array}{rcrcrcrcrcrcrcl} 13A & + & 23B & & & & & & & - & Z & = & 0 \\ \hline 5A & + & 15B & + & S_C & & & & & & & = & 480 \\ 4A & + & 4B & & & + & S_H & & & & & = & 160 \\ 35A & + & 20B & & & & & + & S_M & & & = & 1190 \\ A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0 \end{array} \end{array}$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Substitute: $B = 1/15\,(480 - 5A - S_C)$

$$\begin{array}{l} \max\ Z \text{ subject to} \\ \begin{array}{rcrcrcrcrcrcl} \frac{16}{3}A & & & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\ \hline \frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & & & = & 32 \\ \frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & & & = & 32 \\ \frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & & & = & 550 \\ A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0 \end{array} \end{array}$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Simplex Algorithm: Pivot 1

$$\begin{array}{l} \max\ Z \text{ subject to} \\ \begin{array}{rcrcrcrcrcrcrcl} 13A & + & 23B & & & & & & & - & Z & = & 0 \\ \hline 5A & + & 15B & + & S_C & & & & & & & = & 480 \\ 4A & + & 4B & & & + & S_H & & & & & = & 160 \\ 35A & + & 20B & & & & & + & S_M & & & = & 1190 \\ A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0 \end{array} \end{array}$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Q. Why pivot on column 2 (or 1)?
A. Each unit increase in B increases objective value by \$23.

Q. Why pivot on row 2?
A. Preserves feasibility by ensuring RHS $\geq$ 0.

min ratio rule: $\min\{\ 480/15,\ 160/4,\ 1190/20\ \}$

Simplex Algorithm: Pivot 2

$$\begin{array}{l} \max\ Z \text{ subject to} \\ \begin{array}{rcrcrcrcrcrcl} \frac{16}{3}A & & & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\ \hline \frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & & & = & 32 \\ \frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & & & = & 32 \\ \frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & & & = & 550 \\ A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0 \end{array} \end{array}$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Substitute: $A = 3/8\,(32 + 4/15\,S_C - S_H)$

$$\begin{array}{l} \max\ Z \text{ subject to} \\ \begin{array}{rcrcrcrcrcrcl} & & & - & S_C & - & 2\,S_H & & & - & Z & = & -800 \\ \hline & & B & + & \frac{1}{10}S_C & + & \frac{1}{8}S_H & & & & & = & 28 \\ A & & & - & \frac{1}{10}S_C & + & \frac{3}{8}S_H & & & & & = & 12 \\ & & & - & \frac{25}{6}S_C & - & \frac{85}{8}S_H & + & S_M & & & = & 110 \\ A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0 \end{array} \end{array}$$

Basis = $\{A, B, S_M\}$
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex Algorithm: Optimality

Q. When to stop pivoting?
A. When all coefficients in top row are nonpositive.

Q. Why is resulting solution optimal?
A. Any feasible solution satisfies system of equations in tableaux.

- In particular: $Z = 800 - S_C - 2\,S_H,\ S_C \geq 0,\ S_H \geq 0.$
- Thus, optimal objective value $Z^* \leq 800$.
- Current BFS has value $800 \Rightarrow$ optimal.

$$\begin{array}{l} \max\ Z \text{ subject to} \\ \begin{array}{rcrcrcrcrcrcl} & & & - & S_C & - & 2\,S_H & & & - & Z & = & -800 \\ \hline & & B & + & \frac{1}{10}S_C & + & \frac{1}{8}S_H & & & & & = & 28 \\ A & & & - & \frac{1}{10}S_C & + & \frac{3}{8}S_H & & & & & = & 12 \\ & & & - & \frac{25}{6}S_C & - & \frac{85}{8}S_H & + & S_M & & & = & 110 \\ A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0 \end{array} \end{array}$$

Basis = $\{A, B, S_M\}$
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex Tableaux: Matrix Form

Initial simplex tableaux.

$$\begin{array}{rcrcl} c_B^T\,x_B & + & c_N^T\,x_N & = & Z \\ A_B\,x_B & + & A_N\,x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

Simplex tableaux corresponding to basis B.

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N)\,x_N & = & Z - c_B^T A_B^{-1} b \\ I\,x_B & + & A_B^{-1} A_N\,x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

← subtract $c_B^T A_B^{-1}$ times constraints
← multiply by A_B^{-1}

$x_B = A_B^{-1} b \geq 0$
$x_N = 0$
basic feasible solution

$c_N^T - c_B^T A_B^{-1} A_N \leq 0$
optimal basis

Simplex Algorithm: Corner Cases

Simplex algorithm. Missing details for corner cases.

Q. What if min ratio test fails?
Q. How to find initial basis?
Q. How to guarantee termination?

Unboundedness

Q. What happens if min ratio test fails?

all coefficients in entering column are nonpositive

$$
\begin{array}{l}
\max\ Z \text{ subject to} \\
\begin{array}{cccccccccc}
 & & & + & 2x_4 & + & 20x_5 & - Z & = & 2 \\
x_1 & & & - & 4x_4 & - & 8x_5 & & = & 3 \\
 & x_2 & & + & 5x_4 & - & 12x_5 & & = & 4 \\
 & & x_3 & & & & & & = & 5 \\
x_1, & x_2, & x_3, & & x_4, & & x_5 & & \geq & 0
\end{array}
\end{array}
$$

A. Unbounded objective function.

$$
Z = 2 + 20x_5 \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 + 8x_5 \\ 4 + 12x_5 \\ 5 \\ 0 \\ 0 \end{bmatrix}
$$

Phase I Simplex

Q. How to find initial basis?

$$
\begin{array}{llrcl}
(P) & \max & c^T x & & \\
 & \text{s. t.} & Ax & = & b \\
 & & x & \geq & 0
\end{array}
$$

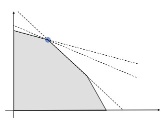
A. Solve (P'), starting from basis consisting of all the z_i variables.

$$
\begin{array}{llrcl}
(P') & \min & \sum_{i=1}^{m} z_i & & \\
 & \text{s. t.} & Ax + Iz & = & b \\
 & & x, \ z & \geq & 0
\end{array}
$$

- Case 1: min > 0 $\Rightarrow$ (P) is infeasible.
- Case 2: min $= 0$, basis has no z_i variables $\Rightarrow$ OK to start Phase II.
- Case 3a: min $= 0$, basis has z_i variables. Pivot z_i variables out of basis. If successful, start Phase II; else remove linear dependent rows.

Simplex Algorithm: Degeneracy

Degeneracy. New basis, same vertex.

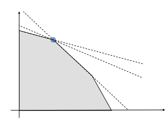


Degenerate pivot. Min ratio = 0.

$$
\begin{array}{l}
\max\ Z \text{ subject to} \\
\begin{array}{cccccccccccccccc}
 & & & & \frac{3}{4}x_4 & - & 20x_5 & + & \frac{1}{2}x_6 & - & 6x_7 & - Z & = & 0 \\
x_1 & & & + & \frac{1}{4}x_4 & - & 8x_5 & - & x_6 & + & 9x_7 & & = & 0 \\
 & x_2 & & + & \frac{1}{2}x_4 & - & 12x_5 & - & \frac{1}{2}x_6 & + & 3x_7 & & = & 0 \\
 & & x_3 & & & & & + & x_6 & & & & = & 1 \\
x_1, & x_2, & x_3, & & x_4, & & x_5, & & x_6, & & x_7 & & \geq & 0
\end{array}
\end{array}
$$

Simplex Algorithm: Degeneracy

Degeneracy. New basis, same vertex.



Cycling. Infinite loop by cycling through different bases that all correspond to same vertex.

Anti-cycling rules.
- Bland's rule: choose eligible variable with smallest index.
- Random rule: choose eligible variable uniformly at random.
- Lexicographic rule: perturb constraints so nondegenerate.

Lexicographic Rule

Intuition. No degeneracy $\Rightarrow$ no cycling.

Perturbed problem.

$$
\begin{array}{llrcl}
(P') & \max & c^T x & & \\
 & \text{s. t.} & Ax & = & b + \varepsilon \\
 & & x & \geq & 0
\end{array}
\qquad \text{where } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \text{such that } \varepsilon_1 \succ \varepsilon_2 \succ \cdots \succ \varepsilon_n
$$

much much greater, say $\epsilon_i = \delta^i$ for small δ

Lexicographic rule. Apply perturbation virtually by manipulating ϵ symbolically:

$$
17 + 5\varepsilon_1 + 11\varepsilon_2 + 8\varepsilon_3 \leq 17 + 5\varepsilon_1 + 14\varepsilon_2 + 3\varepsilon_3
$$

Lexicographic Rule

Intuition. No degeneracy $\Rightarrow$ no cycling.

Perturbed problem.

$$
\begin{array}{llrcl}
(P') & \max & c^T x & & \\
 & \text{s. t.} & Ax & = & b + \varepsilon \\
 & & x & \geq & 0
\end{array}
\qquad \text{where } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \text{such that } \varepsilon_1 \succ \varepsilon_2 \succ \cdots \succ \varepsilon_n
$$

much much greater, say $\epsilon_i = \delta^i$ for small δ

Claim. In perturbed problem, $x_B = A_B^{-1}(b + \epsilon)$ is always nonzero.

Pf. The j^{th} component of x_B is a (nonzero) linear combination of the components of $b + \epsilon$ $\Rightarrow$ contains at least one of the ϵ_i terms.

which can't cancel

Corollary. No cycling.

Simplex Algorithm: Practice

Remarkable property. In practice, simplex algorithm typically terminates after at most $2(m+n)$ pivots.

but no polynomial pivot rule known

Issues.
- Choose the pivot.
- Maintain sparsity.
- Ensure numerical stability.
- Preprocess to eliminate variables and constraints.

Commercial solvers can solve LPs with millions of variables and tens of thousands of constraints.

LP duality

LP Duality

Primal problem.

$$\begin{array}{llrcrcr} (P) & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \leq & 480 \\ & & 4A & + & 4B & \leq & 160 \\ & & 35A & + & 20B & \leq & 1190 \\ & & A & , & B & \geq & 0 \end{array}$$

Goal. Find a lower bound on optimal value.

Easy. Any feasible solution provides one.

Ex 1. $(A, B) = (34, 0) \Rightarrow z^* \geq 442$
Ex 2. $(A, B) = (0, 32) \Rightarrow z^* \geq 736$
Ex 3. $(A, B) = (7.5, 29.5) \Rightarrow z^* \geq 776$
Ex 4. $(A, B) = (12, 28) \Rightarrow z^* \geq 800$

LP Duality

Primal problem.

$$\begin{array}{llrcrcr} (P) & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \leq & 480 \\ & & 4A & + & 4B & \leq & 160 \\ & & 35A & + & 20B & \leq & 1190 \\ & & A & , & B & \geq & 0 \end{array}$$

Goal. Find an upper bound on optimal value.

Ex 1. Multiply 2nd inequality by 6: $24A + 24B \leq 960$.

$$\Rightarrow \quad z^* = \underbrace{13A + 23B}_{\text{objective function}} \leq 24A + 24B \leq 960.$$

LP Duality

Primal problem.

$$\begin{array}{llrcrcr} (P) & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \leq & 480 \\ & & 4A & + & 4B & \leq & 160 \\ & & 35A & + & 20B & \leq & 1190 \\ & & A & , & B & \geq & 0 \end{array}$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 2 times 1st inequality to 2nd inequality:

$$\Rightarrow \quad z^* = 13A + 23B \leq 14A + 34B \leq 1120.$$

LP Duality

Primal problem.

$$\begin{array}{llrcrcr} (P) & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \leq & 480 \\ & & 4A & + & 4B & \leq & 160 \\ & & 35A & + & 20B & \leq & 1190 \\ & & A & , & B & \geq & 0 \end{array}$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 1 times 1st inequality to 2 times 2nd inequality:

$$\Rightarrow \quad z^* = 13A + 23B \leq 13A + 23B \leq 800.$$

Recall lower bound. $(A, B) = (12, 28) \Rightarrow z^* \geq 800$
Combine upper and lower bounds: $z^* = 800$.

LP Duality

Primal problem.

$$
\begin{array}{lrcrcl}
\text{(P)} \quad \max & 13A & + & 23B & & \\
\text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

Idea. Add nonnegative combination (C, H, M) of the constraints s.t.

$$
\begin{array}{rl}
13A+23B & \leq \; (5C+4H+35M)\; A \; + \; (15C+4H+20M)\; B \\
 & \leq \; 480C+160H+1190M
\end{array}
$$

Dual problem. Find best such upper bound.

$$
\begin{array}{lrcrcrcl}
\text{(D)} \quad \min & 480C & + & 160H & + & 1190M & & \\
\text{s. t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
 & 15C & + & 4H & + & 20M & \geq & 23 \\
 & C & , & H & , & M & \geq & 0
\end{array}
$$

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LP Duality: Economic Interpretation

Brewer: find optimal mix of beer and ale to maximize profits.

$$
\begin{array}{lrcrcl}
\text{(P)} \quad \max & 13A & + & 23B & & \\
\text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

Entrepreneur: buy individual resources from brewer at min cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if $5C + 4H + 35M < 13$.

$$
\begin{array}{lrcrcrcl}
\text{(D)} \quad \min & 480C & + & 160H & + & 1190M & & \\
\text{s. t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
 & 15C & + & 4H & + & 20M & \geq & 23 \\
 & C & , & H & , & M & \geq & 0
\end{array}
$$

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LP Duals

Canonical form.

$$
\begin{array}{lrcl}
\text{(P)} \quad \max\, c^T x & & & \\
\text{s. t.} & Ax & \leq & b \\
 & x & \geq & 0
\end{array}
\qquad
\begin{array}{lrcl}
\text{(D)} \quad \min\, y^T b & & & \\
\text{s. t.} & A^T y & \geq & c \\
 & y & \geq & 0
\end{array}
$$

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Double Dual

Canonical form.

$$
\begin{array}{lrcl}
\text{(P)} \quad \max\, c^T x & & & \\
\text{s. t.} & Ax & \leq & b \\
 & x & \geq & 0
\end{array}
\qquad
\begin{array}{lrcl}
\text{(D)} \quad \min\, y^T b & & & \\
\text{s. t.} & A^T y & \geq & c \\
 & y & \geq & 0
\end{array}
$$

Property. The dual of the dual is the primal.
Pf. Rewrite (D) as a maximization problem in canonical form; take dual.

$$
\begin{array}{lrcl}
\text{(D')} \quad \max & -y^T b & & \\
\text{s. t.} & -A^T y & \leq & -c \\
 & y & \geq & 0
\end{array}
\qquad
\begin{array}{lrcl}
\text{(DD)} \quad \min & -c^T z & & \\
\text{s. t.} & -(A^T)^T z & \geq & -b \\
 & z & \geq & 0
\end{array}
$$

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Taking Duals

LP dual recipe.

Primal (P)	maximize	minimize	Dual (D)
constraints	$a x = b_i$ $a x \leq b$ $a x \geq b_i$	y_i unrestricted $y_i \geq 0$ $y_i \leq 0$	variables
variables	$x_j \leq 0$ $x_j \geq 0$ unrestricted	$a^T y \geq c_j$ $a^T y \leq c_j$ $a^T y = c_j$	constraints

Pf. Rewrite LP in standard form and take dual.

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Strong duality

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LP Strong Duality

Theorem. [Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947]
For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, if (P) and (D) are nonempty, then max = min.

$$\text{(P)} \quad \max\ c^T x \quad \text{s. t.} \quad Ax \leq b, \quad x \geq 0$$

$$\text{(D)} \quad \min\ y^T b \quad \text{s. t.} \quad A^T y \geq c, \quad y \geq 0$$

Generalizes:
- Dilworth's theorem.
- König-Egervary theorem.
- Max-flow min-cut theorem.
- von Neumann's minimax theorem.
- …

Pf. [ahead]