CS21
Decidability and Tractability

Lecture 2

January 9, 2008

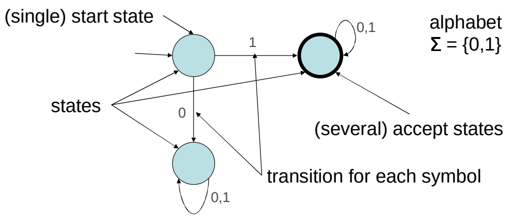
Outline

- Finite Automata
- Nondeterministic Finite Automata
- Closure under regular operations
- NFA, FA equivalence

Terminology

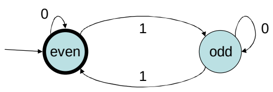
- finite alphabet Σ : a set of symbols
- language L ⊆ Σ^* : subset of strings over Σ
- a machine takes an input string and either
  - accepts, rejects, or
  - loops forever
- a machine recognizes the set of strings that lead to accept
- a machine decides a language L if it accepts x ∈ L and rejects x ∉ L

FA diagrams



- read input one symbol at a time; follow arrows; accept if end in accept state

Example FA



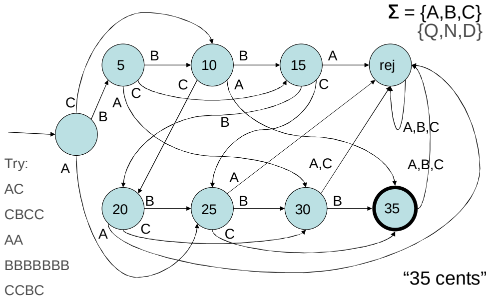
- What language does this FA recognize?

  L = {x : x ∈ {0,1}*, x has even # of 1s}
- illustrates fundamental feature/limitation of FA:
  - "tiny" memory
  - in this example only "remembers" 1 bit of info.

Example FA

Σ = {A,B,C}

{Q,N,D}



Try:

AC

CBCC

AA

BBBBBBB

CCBC

"35 cents"

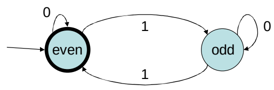
FA formal definition

A finite automaton is a 5-tuple

$(Q, \Sigma, \delta, q_0, F)$

- Q is a finite set called the states
- Σ is a finite set called the alphabet
- $\delta: Q \times \Sigma \to Q$ is a function called the transition function
- q_0 is an element of Q called the start state
- F is a subset of Q called the accept states

FA formal definition



- Specification of this FA in formal terms:
  - $Q = \{\text{even}, \text{odd}\}$
  - $\Sigma = \{0,1\}$
  - $q_0 = \text{even}$
  - $F = \{\text{even}\}$

function δ:

$\delta(\text{even}, 0) = \text{even}$

$\delta(\text{even}, 1) = \text{odd}$

$\delta(\text{odd}, 0) = \text{odd}$

$\delta(\text{odd}, 1) = \text{even}$

Formal description of FA operation

finite automaton

$M = (Q, \Sigma, \delta, q_0, F)$

accepts a string

$w = w_1 w_2 w_3 \ldots w_n \in \Sigma^*$

if $\exists$ a sequence $r_0, r_1, r_2, \ldots, r_n$ of states for which

- $r_0 = q_0$
- $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0,1,2, \ldots, n-1$
- $r_n \in F$

What now?

- We have a model of computation

  (Maybe this is it. Maybe everything we can do with real computers we can do with FA…)

- try to characterize the languages FAs can recognize
  - investigate closure under certain operations
- show that some languages not of this type

Characterizing FA languages

- We will show that the set of languages recognized by FA is closed under:
  - union "$C = (A \cup B)$"
  - concatenation "$C = (A \circ B)$"
  - star "$C = A^*$"
- Meaning: if A and B are languages recognized by a FA, then C is a language recognized by a FA

Characterizing FA languages

- union "$C = (A \cup B)$"

  $(A \cup B) = \{x : x \in A \text{ or } x \in B \text{ or both}\}$

- concatenation "$C = (A \circ B)$"

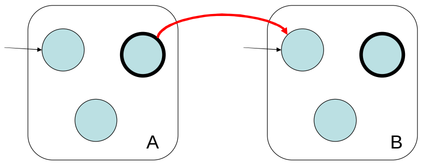
  $(A \circ B) = \{xy : x \in A \text{ and } y \in B\}$

- star "$C = A^*$" (note: **ε** always in A^*)

  $A^* = \{x_1 x_2 x_3 \ldots x_k : k \geq 0 \text{ and each } x_i \in A\}$

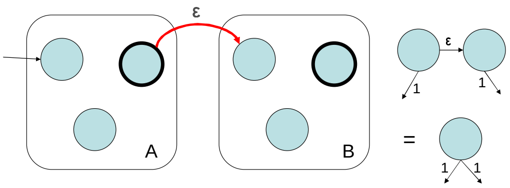
Concatenation attempt

(A ° B) = {xy : x ∈ A and y ∈ B}



What label do we put on the new transition?

Concatenation attempt

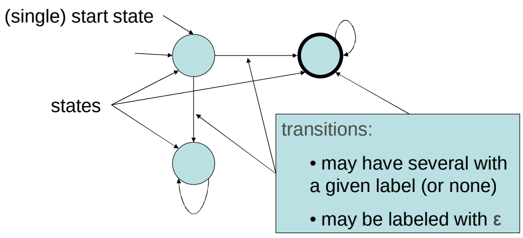


- Need it to happen "for free": label with **ε** (?)
- allows construct with multiple transitions with the same label (!?)

Nondeterministic FA

- We will make life easier by describing an additional feature (nondeterminism) that helps us to "program" FAs
- We will prove that FAs with this new feature can be simulated by ordinary FA
  - same spirit as programming constructs like procedures
- The concept of nondeterminism has a significant role in TCS and this course.

NFA diagrams

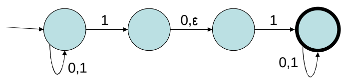


- At each step, several choices for next state

NFA operation

- Example of NFA operation:

alphabet
Σ = {0,1}



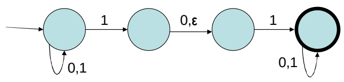
input: 0 1 0

not accepted

NFA operation

- Example of NFA operation:

alphabet
Σ = {0,1}



input: 1 1 0

accepted

NFA operation

- One way to think of NFA operation:

- string $x = x_1x_2x_3\ldots x_n$ accepted if and only if
  - there exists a way of inserting ε's into x

    $x_1\varepsilon\varepsilon x_2\, x_3\ldots\varepsilon x_n$
  - so that there exists a path of transitions from the start state to an accept state

NFA formal definition

A nondeterministic FA

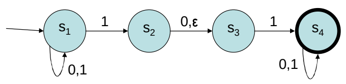
(Q, Σ, δ

transit
labeled
alpha
symbols or ε

"powerset of Q": the set of all subsets of Q

- Q is a finite set called
- Σ is a finite set called the alphabet
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \wp(Q)$ is a function called the transition function
- q_0 is an element of Q called the start state
- F is a subset of Q called the accept states

NFA formal definition



- Specification of this NFA in formal terms:
  - $Q = \{s_1, s_2, s_3, s_4\}$
  - $\Sigma = \{0,1\}$
  - $q_0 = s_1$
  - $F = \{s_4\}$

$\delta(s_1, 0) = \{s_1\}$
$\delta(s_1, 1) = \{s_1, s_2\}$
$\delta(s_1, \varepsilon) = \{\ \}$
$\delta(s_2, 0) = \{s_3\}$
$\delta(s_2, 1) = \{\}$
$\delta(s_2, \varepsilon) = \{s_3\}$
$\delta(s_3, 0) = \{\}$
$\delta(s_3, 1) = \{s_4\}$
$\delta(s_3, \varepsilon) = \{\ \}$
$\delta(s_4, 0) = \{s_4\}$
$\delta(s_4, 1) = \{s_4\}$
$\delta(s_4, \varepsilon) = \{\}$

Formal description of NFA operation

NFA $M = (Q, \Sigma, \delta, q_0, F)$

accepts a string $w = w_1w_2w_3\ldots w_n \in \Sigma^*$

if w can be written (by inserting ε's) as:

$$y = y_1y_2y_3\ldots y_m \in (\Sigma \cup \{\varepsilon\})^*$$

and ∃ sequence $r_0, r_1, \ldots, r_m$ of states for which

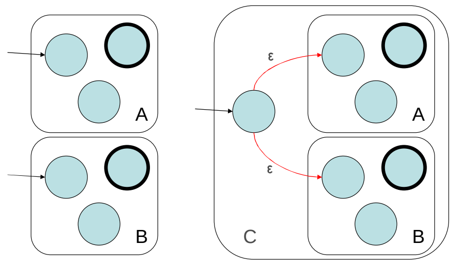
- $r_0 = q_0$
- $r_{i+1} \in \delta(r_i, y_{i+1})$ for i = 0,1,2, …, m-1
- $r_m \in F$

Closures

- Recall: want to show the set of languages recognized by NFA is closed under:
  - union "C = (A ∪ B)"
  - concatenation "C = (A ∘ B)"
  - star " C = A* "

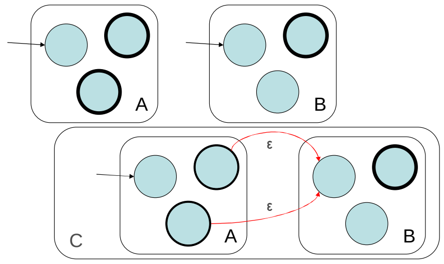
Closure under union

$C = (A \cup B) = \{x : x \in A \text{ or } x \in B\}$



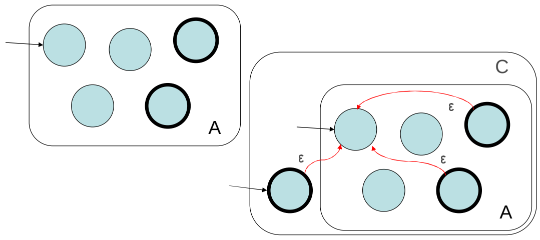
Closure under concatenation

C = (A ° B) = {xy : x ∈ A and y ∈ B}



Closure under star

$C = A^* = \{x_1x_2x_3\ldots x_k : k \geq 0 \text{ and each } x_i \in A\}$



NFA, FA equivalence

Theorem: a language L is recognized by a FA if and only if L is recognized by a NFA.

Must prove *two* directions:

(⇒) L is recognized by a FA implies L is recognized by a NFA.

(⇐) L is recognized by a NFA implies L is recognized by a FA.

(usually one is easy, the other more difficult)

NFA, FA equivalence

(⇒) L is recognized by a FA implies L is recognized by a NFA

Proof: a finite automaton *is* a nondeterministic finite automaton that happens to have no **ε**-transitions, and for which each state has exactly one outgoing transition for each symbol.

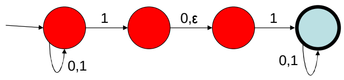
NFA, FA equivalence

(⇐) L is recognized by a NFA implies L is recognized by a FA.

Proof: we will build a FA that *simulates* the NFA (and thus recognizes the same language).

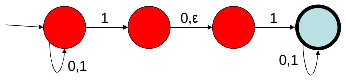
- alphabet will be the same
- what are the states of the FA?

NFA, FA equivalence



- given NFA $M = (Q, \Sigma, \delta, q_0, F)$
- construct FA $M' = (Q', \Sigma', \delta', q_0', F')$
- same alphabet: $\Sigma' = \Sigma$
- states are subsets of M's states: $Q' = \wp(Q)$
- if we are in state $R \in Q'$ and we read symbol $a \in \Sigma'$, what is the new state?

NFA, FA equivalence



– given NFA M = (Q, Σ, δ, q_0, F)

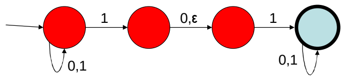
– construct FA M' = (Q', Σ', δ', q_0', F')

Helpful def'n: E(S) = {q ∈ Q : q reachable from S by traveling along 0 or more **ε**-transitions}

– new transition fn: δ'(R, a) = $\cup_{r\in R}$ E(δ(r, a))

= "all nodes reachable from R by following an a-transition, and then 0 or more **ε**-transitions"

NFA, FA equivalence



– given NFA M = (Q, Σ, δ, q_0, F)

– construct FA M' = (Q', Σ', δ', q_0', F')

– new start state: q_0' = E({q_0})

– new accept states:

F' = {R ∈ Q' : R contains an accept state of M)

NFA, FA equivalence

- We have proved (⇐) by construction.

Formally we should also prove that the construction works, by induction on the number of steps of the computation.

– at each step, the state of the FA M' is exactly the set of reachable states of the NFA M…

So far…

Theorem: the set of languages recognized by NFA is closed under union, concatenation, and star.

Theorem: a language L is recognized by a FA if and only if L is recognized by a NFA.

Theorem: the set of languages recognized by FA is closed under union, concatenation, and star.

Coming up…

- Describe the set of languages that can be built up from:
  - unions
  - concatenations
  - star operations
- Called "patterns" or regular expressions
- **Theorem**: a language L is recognized by a FA if and only if L is described by a regular expression.