CS21
Decidability and Tractability

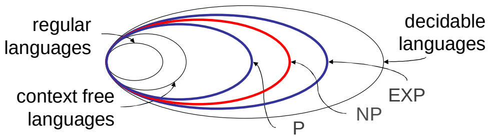
Lecture 19

February 22, 2008

Outline

- NP-complete problems…
  - independent set
  - vertex cover
  - clique
  - Hamilton path/cycle
  - Traveling Salesperson Problem

The class NP



Definition: NTIME(t(n)) = {L : there exists a NTM M that decides L in time O(t(n))}

$$NP = \cup_{k \geq 1} NTIME(n^k)$$

Poly-time verifiers

- NP = {L : L decide e NTM}

“witness” or “certificate”

- Very useful alternate definition

efficiently verifiable

Theorem: language L is in NP if it is expressible as:

$$L = \{ x \mid \exists y, |y| \leq |x|^k, (x, y) \in R \}$$

where R is a language in P.

- poly-time TM M_R deciding R is a “verifier”

Search vs. Decision

- Definition: given a graph G = (V, E), an independent set in G is a subset V’ ⊆ V such that for all u,w ∈ V’ (u,w) ∉ E
- A problem:

  given G, find the largest independent set
- This is called a search problem
  - searching for *optimal* object of some type
  - comes up frequently

Search vs. Decision

- We want to talk about languages (or decision problems)
- Most search problems have a natural, related decision problem by adding a bound “k”; for example:
  - search problem: given G, find the largest independent set
  - decision problem: given (G, k), is there an independent set of size *at least* k

Ind. Set is NP-complete

Theorem: the following language is NP-complete:

IS = {(G, k) : G has an IS of size $\geq$ k}.

- Proof:
  - Part 1: IS $\in$ NP. Proof?
  - Part 2: IS is NP-hard.
    - reduce from 3-SAT

Ind. Set is NP-complete

- We are reducing from the language:

3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

to the language:

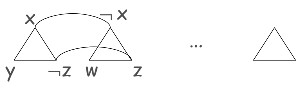
IS = {(G, k) : G has an IS of size $\geq$ k}.

Ind. Set is NP-complete

The reduction f: given

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

we produce graph G_φ:

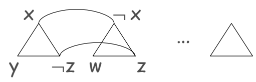


- one triangle for each of m clauses
- edge between every pair of contradictory literals
- set k = m

Ind. Set is NP-complete

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

f(φ) =
(G, # clauses)

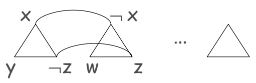


- Is f poly-time computable?
- YES maps to YES?
  - 1 true literal per clause in satisfying assign. A
  - choose corresponding vertices (1 per triangle)
  - IS, since no contradictory literals in A

Ind. Set is NP-complete

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

f(φ) =
(G, # clauses)



- NO maps to NO?
  - IS can have at most 1 vertex per triangle
  - IS of size $\geq$ # clauses must have exactly 1 per
  - since IS, no contradictory vertices
  - can produce satisfying assignment by setting these literals to true

Vertex cover

- Definition: given a graph G = (V, E), a vertex cover in G is a subset V' $\subseteq$ V such that for all (u,w) $\in$ E, u $\in$ V' or w $\in$ V'
- A search problem:

  given G, find the smallest vertex cover

- corresponding language (decision problem):

  VC = {(G, k) : G has a VC of size $\leq$ k}.

Vertex Cover is NP-complete

Theorem: the following language is NP-complete:

VC = {(G, k) : G has a VC of size ≤ k}.

- Proof:
  - Part 1: VC ∈ NP. Proof?
  - Part 2: VC is NP-hard.
    - reduce from?

Vertex Cover is NP-complete

- We are reducing from the language:

IS = {(G, k) : G has an IS of size ≥ k}

to the language:

VC = {(G, k) : G has a VC of size ≤ k}.

Vertex Cover is NP-complete

- How are IS, VC related?
- Given a graph G = (V, E) with n nodes
  - if V' ⊆ V is an independent set of size k
  - then V-V' is a vertex cover of size n-k
- Proof:
  - suppose not. Then there is some edge with neither endpoint in V-V'. But then both endpoints are in V'. contradiction.

Vertex Cover is NP-complete

- How are IS, VC related?
- Given a graph G = (V, E) with n nodes
  - if V' ⊆ V is a vertex cover of size k
  - then V-V' is an independent set of size n-k
- Proof:
  - suppose not. Then there is some edge with both endpoints in V-V'. But then neither endpoint is in V'. contradiction.

Vertex Cover is NP-complete

The reduction:

- given an instance of IS: (G, k) f produces the pair (G, n-k)

- f poly-time computable?
- YES maps to YES?
  - IS of size ≥ k in G ⇒ VC of size ≤ n-k in G
- NO maps to NO?
  - VC of size ≤ n-k in G ⇒ IS of size ≥ k in G

Clique

- Definition: given a graph G = (V, E), a clique in G is a subset V'⊆ V such that for all u,v ∈ V', (u, v) ∈ E
- A search problem:

  given G, find the largest clique

- corresponding language (decision problem):

  CLIQUE = {(G, k) : G has a clique of size ≥ k}.

Clique is NP-complete

Theorem: the following language is NP-complete:

CLIQUE = {(G, k) : G has a clique of size ≥ k}

- Proof:
  – Part 1: CLIQUE ∈ NP. Proof?
  – Part 2: CLIQUE is NP-hard.
    - reduce from?

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Clique is NP-complete

- We are reducing from the language:

IS = {(G, k) : G has an IS of size ≥ k}

to the language:

CLIQUE = {(G, k) : G has a CLIQUE of size ≥ k}.

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Clique is NP-complete

- How are IS, CLIQUE related?
- Given a graph G = (V, E), define its complement G' = (V, E' = {(u,v) : (u,v) ∉ E})
  – if V' ⊆ V is an independent set in G of size k
  – then V' is a clique in G' of size k
- Proof:
  – *Every* pair of vertices u,v ∈ V' has no edge between them in G. Therefore they have an edge between them in G'.

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Clique is NP-complete

- How are IS, CLIQUE related?
- Given a graph G = (V, E), define its complement G' = (V, E' = {(u,v) : (u,v) ∉ E})
  – if V' ⊆ V is a clique in G' of size k
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- Proof:
  – *Every* pair of vertices u,v ∈ V' has an edge between them in G'. Therefore they have no edge between them in G.

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Clique is NP-complete

The reduction:

– given an instance of IS: (G, k) f produces the pair (G', k)

- f poly-time computable?
- YES maps to YES?
  – IS of size ≥ k in G ⇒ CLIQUE of size ≥ k in G'
- NO maps to NO?
  – CLIQUE of size ≥ k in G' ⇒ IS of size ≥ k in G

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Hamilton Path

- Definition: given a directed graph G = (V, E), a Hamilton path in G is a directed path that touches every node exactly once.
- A language (decision problem):

HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

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HAMPATH is NP-complete

Theorem: the following language is NP-complete:

HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

- Proof:
  - Part 1: HAMPATH ∈ NP. Proof?
  - Part 2: HAMPATH is NP-hard.
    - reduce from?

HAMPATH is NP-complete

- We are reducing from the language:

3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

to the language:

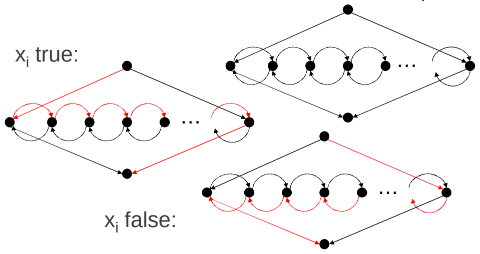
HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

HAMPATH is NP-complete

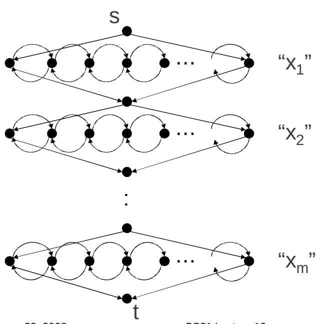
- We want to construct a graph from φ with the following properties:
  - a satisfying assignment to φ translates into a Hamilton Path from s to t
  - a Hamilton Path from s to t can be translated into a satisfying assignment for φ
- We will build the graph up from pieces called gadgets that “simulate” the clauses and variables of φ.

HAMPATH is NP-complete

- The variable gadget (one for each x_i):



HAMPATH is NP-complete



- path from s to t translates into a truth assignment to $x_1 \ldots x_m$
- why must the path be of this form?

HAMPATH is NP-complete

$\varphi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_4 \lor x_3) \land \ldots \land (\ldots)$

- How to ensure that all k clauses are satisfied?
- need to add nodes
  - can be visited in path if the clause is satisfied
  - if visited in path, implies clause is satisfied by the assignment given by path through variable gadgets