CS21
Decidability and Tractability

Lecture 12

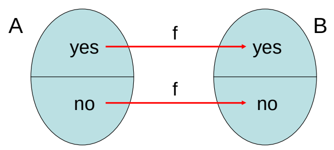
February 4, 2008

Outline

- more undecidable problems
  - computation histories
  - suprising contrasts between decidable/undecidable
  - Rice's Theorem
  - Post Correspondence Problem
- recursion theorem

Definition of reduction

- More refined notion of reduction:
  - "many-one" reduction (commonly)
  - "mapping" reduction (book)



reduction from language A to language B

Dec. and undec. problems

- two problems regarding Context-Free Grammars:
  - does a CFG generate all strings:

    ALL_{CFG} = {<G> : G is a CFG and L(G) = Σ*}
  - CFG emptiness:

    E_{CFG} = {<G> : G is a CFG and L(G) = Ø}
- Both decidable? both undecidable? one decidable?

Dec. and undec. problems

<u>Theorem</u>: ALL_{CFG} is undecidable.

Proof:
- reduce from co-A_{TM} (i.e. show co-$A_{TM} \leq_m ALL_{CFG}$)
- what should f(<M, w>) produce?
- Idea:
  - produce CFG G that generates all strings that are ***not*** accepting computation histories of M on w

Dec. and undec. problems

Proof:
- build a NPDA, then convert to CFG
- want to accept strings not of this form,

  $\#C_1\#C_2\#C_3\#...\#C_k\#$

  plus strings of this form but where
  - C_1 is not the start config. of M on input w, or
  - C_k is not an accept. config. of M on input w, or
  - C_i does not yield in one step C_{i+1} for some i

Dec. and undec. problems

Proof:

- our NPDA nondeterministically checks one of:
  - C_1 is not the start config. of M on input w, or
  - C_k is not an accept. config. of M on input w, or
  - C_i does not yield in one step C_{i+1} for some I
  - input has fewer than two #'s
- details of first two?
- to check third condition:
  - nondeterministically guess C_i starting position
  - how to check that C_i doesn't yield in 1 step C_{i+1} ?

Dec. and undec. problems

Proof:

- checking:
  - C_i does not yield in one step C_{i+1} for some i
- push C_i onto stack
- at #, start popping C_i and compare to C_{i+1}
  - accept if mismatch away from head location, or
  - symbols around head changed in a way inconsistent with M's transition function.
- is everything described possible with NPDA?

Dec. and undec. problems

Proof:

- Problem: cannot compare C_i to C_{i+1}
- could prove in same way that proved
  $\{ww: w \in \Sigma^*\}$ not context-free
- recall that
  $\{ww^R: w \in \Sigma^*\}$ is context-free
- free to tweak construction of G in the reduction
- solution: write computation history:
  $\#C_1\#C_2^R\#C_3\#C_4^R...\#C_k\#$

Dec. and undec. problems

Proof:

- f(<M, w>) = <G> equiv. to NPDA below:

on input x, accept if not of form:

$\#C_1\#C_2^R\#C_3\#C_4^R...\#C_k\#$

- accept if C_1 is the not the start configuration for M on input w
- accept if check that C_i does not yield in one step C_{i+1}
- accept if C_k is not an accepting configuration for M

- is f computable?
- YES maps to YES?
  $\langle M, w\rangle \in$ co-$A_{TM} \Rightarrow f(M, w) \in ALL_{CFG}$
- NO maps to NO?
  $\langle M, w\rangle \notin$ co-$A_{TM} \Rightarrow f(M, w) \notin ALL_{CFG}$

Rice's Theorem

- We have seen that the following properties of TM's are undecidable:
  - TM accepts string w
  - TM halts
  - TM accepts the empty language
  - TM accepts a regular language
- Can we describe a single generic reduction for all these proofs?
- Yes. *Every* property of TMs undecidable!

Rice's Theorem

- A TM property is a language P for which
  - if $L(M_1) = L(M_2)$ then $\langle M_1\rangle \in P$ iff $\langle M_2\rangle \in P$
- TM property P is nontrivial if
  - there exists a TM M_1 for which $\langle M_1\rangle \in P$, and
  - there exists a TM M_2 for which $\langle M_2\rangle \notin P$.

<u>Rice's Theorem</u>: Every nontrivial TM property is undecidable.

Rice's Theorem

- The setup:
  - let $T_\emptyset$ be a TM for which $L(T_\emptyset) = \emptyset$
    - technicality: if $<T_\emptyset> \in P$ then work with property co-P instead of P.
    - conclude co-P undecidable; therefore P undec. due to closure under complement
  - so, WLOG, assume $<T_\emptyset> \notin P$
  - non-triviality ensures existence of TM M_1 such that $<M_1> \in P$

Rice's Theorem

Proof:
- reduce from A_{TM} (i.e. show $A_{TM} \leq_m P$)
- what should f(<M, w>) produce?
- f(<M, w>) = <M'> described below:

on input x,
• accept iff M accepts w and M_1 accepts x
(intersection of two RE languages)

- f computable?
- YES maps to YES?

  $<M, w> \in A_{TM} \Rightarrow$
  $L(f(M, w)) = L(M_1) \Rightarrow$
  $f(M, w) \in P$

Rice's Theorem

Proof:
- reduce from A_{TM} (i.e. show $A_{TM} \leq_m P$)
- what should f(<M, w>) produce?
- f(<M, w>) = <M'> described below:

on input x,
• accept iff M accepts w and M_1 accepts x
(intersection of two RE languages)

- NO maps to NO?

  $<M, w> \notin A_{TM} \Rightarrow$
  $L(f(M, w)) = L(T_\emptyset) \Rightarrow$
  $f(M, w) \notin P$

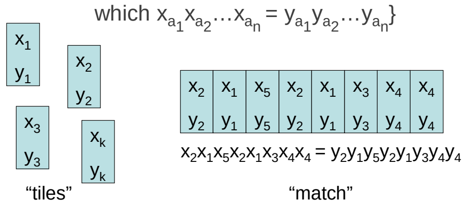
Post Correspondence Problem

- many undecidable problems unrelated to TMs and automata
- classic example: Post Correspondence Problem

$$PCP = \{<(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)> : x_i, y_i \in \Sigma^* \text{ and there exists } (a_1, a_2, \ldots, a_n) \text{ for which } x_{a_1}x_{a_2}\ldots x_{a_n} = y_{a_1}y_{a_2}\ldots y_{a_n}\}$$

Post Correspondence Problem

$$PCP = \{<(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)> : x_i, y_i \in \Sigma^* \text{ and there exists } (a_1, a_2, \ldots, a_n) \text{ for which } x_{a_1}x_{a_2}\ldots x_{a_n} = y_{a_1}y_{a_2}\ldots y_{a_n}\}$$



Post Correspondence Problem

Theorem: PCP is undecidable.

Proof:
- reduce from A_{TM} (i.e. show $A_{TM} \leq_m PCP$)
- two step reduction makes it easier
- first, show $A_{TM} \leq_m MPCP$

  (MPCP = "modified PCP")
- next, show $MPCP \leq_m PCP$

Post Correspondence Problem

MPCP = {<(x_1, y_1), (x_2, y_2), …, (x_k, y_k)> : $x_i, y_i \in \Sigma^*$ and there exists ($a_1, a_2, \ldots, a_n$) for which $x_1 x_{a_1} x_{a_2} \ldots x_{a_n} = y_1 y_{a_1} y_{a_2} \ldots y_{a_n}$}

Proof of MPCP $\leq_m$ PCP:

- notation: for a string $u = u_1 u_2 u_3 \ldots u_m$
  - $*u$ means the string $*u_1*u_2*u_3*u_4\ldots*u_m$
  - $u*$ means the string $u_1*u_2*u_3*u_4\ldots*u_m*$
  - $*u*$ means the string $*u_1*u_2*u_3*u_4\ldots*u_m*$

Post Correspondence Problem

Proof of MPCP $\leq_m$ PCP:

- given an instance (x_1, y_1), …, (x_k, y_k) of MPCP
- produce an instance of PCP:
  - ($*x_1$, $*y_1*$) , ($*x_1$, y_1*), ($*x_2$, y_2*), …, ($*x_k$, y_k*), ($*\blacklozenge$, $\blacklozenge$)
- YES maps to YES?
  - given a match in original MPCP instance, can produce a match in the new PCP instance
- NO maps to NO?
  - given a match in the new PCP instance, can produce a match in the original MPCP instance

Post Correspondence Problem

- YES maps to YES?
  - given a match in original MPCP instance, can produce a match in the new PCP instance

x_1	x_4	x_5	x_2	x_1	x_3	x_4	x_4
y_1	y_4	y_5	y_2	y_1	y_3	y_4	y_4

$*x_1$	$*x_4$	$*x_5$	$*x_2$	$*x_1$	$*x_3$	$*x_4$	$*x_4$	$*\blacklozenge$
$*y_1*$	y_4*	y_5*	y_2*	y_1*	y_3*	y_4*	y_4*	$\blacklozenge$

Post Correspondence Problem

- NO maps to NO?
  - given a match in the new PCP instance, can produce a match in the original MPCP instance

can't match unless start with this tile

$*x_1$	$*x_4$	$*x_5$	$*x_2$	$*x_1$	$*x_3$	$*x_4$	$*x_4$	$*\blacklozenge$
$*y_1*$	y_4*	y_5*	y_2*	y_1*	y_3*	y_4*	y_4*	$\blacklozenge$

x_1	x_4	x_5	x_2	x_1	x_3	x_4	x_4
y_1	y_4	y_5	y_2	y_1	y_3	y_4	y_4

"*" symbols must align

can only appear at the end

Post Correspondence Problem

Theorem: PCP is undecidable.

Proof:

- show $A_{TM} \leq_m$ MPCP

  MPCP = {<(x_1, y_1), (x_2, y_2), …, (x_k, y_k)> : $x_i, y_i \in \Sigma^*$ and there exists ($a_1, a_2, \ldots, a_n$) for which $x_1 x_{a_1} x_{a_2} \ldots x_{a_n} = y_1 y_{a_1} y_{a_2} \ldots y_{a_n}$}

- show MPCP $\leq_m$ PCP

Post Correspondence Problem

Proof of $A_{TM} \leq_m$ MPCP:

- given instance of A_{TM}: <M, w>
- idea: a match will record an accepting computation history for M on input w
- start tile records starting configuration:
  - add tile (#, #$q_0 w_1 w_2 w_3 \ldots w_n$#)

#		#	
#$q_0 w_1 w_2 \ldots w_n$#	=	#C_1#	