

CS21 Decidability and Tractability

Lecture 11
February 1, 2008

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Outline

- reductions
- more undecidable problems
 - computation histories
 - surprising contrasts between decidable/undecidable
 - Rice's Theorem
 - Post Correspondence Problem

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Reductions

- A better option:
 - to prove NEW is decidable, show how to transform it into a known decidable problem OLD so that solution to OLD can be used to solve NEW.
 - to prove NEW is undecidable, show how to transform a known undecidable problem OLD into NEW so that solution to NEW can be used to solve OLD.
- called a **reduction**

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Definition of reduction

- Can you reduce co-HALT to HALT?
- We know that HALT is RE
- Does this show that co-HALT is RE?
 - recall, we showed co-HALT is not RE
- our notion of reduction cannot distinguish complements

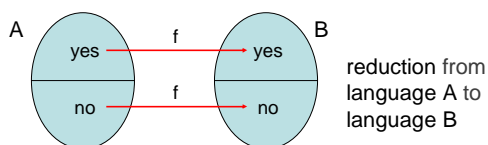
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Definition of reduction

- More refined notion of reduction:
 - “many-one” reduction (commonly)
 - “mapping” reduction (book)

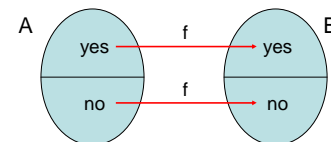


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Definition of reduction



- function f should be computable
- Definition:** $f : \Sigma^* \rightarrow \Sigma^*$ is computable if there exists a TM M_f such that on every $w \in \Sigma^*$ M_f halts on w with $f(w)$ written on its tape.

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Definition of reduction

- Notation: “A many-one reduces to B” is written

$$A \leq_m B$$

- “yes maps to yes and no maps to no” means:
 $w \in A$ maps to $f(w) \in B$ & $w \notin A$ maps to $f(w) \notin B$

- B is at least as “hard” as A
 – more accurate: B at least as “expressive” as A

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Using reductions

Definition: $A \leq_m B$ if there is a computable function f such that for all w

$$w \in A \leftrightarrow f(w) \in B$$

Theorem: if $A \leq_m B$ and B is decidable then A is decidable

Proof:

- decider for A: on input w , compute $f(w)$, run decider for B, do whatever it does.

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Using reductions

- Main use: given language NEW, prove it is undecidable by showing $OLD \leq_m NEW$, where OLD known to be undecidable
 - proof by contradiction
 - if NEW decidable, then OLD decidable
 - OLD undecidable. Contradiction.
- common to reduce in wrong direction.
- review this argument to check yourself.

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Using reductions

Theorem: if $A \leq_m B$ and B is RE then A is RE

Proof:

- TM for recognizing A: on input w , compute $f(w)$, run TM that recognizes B, do whatever it does.
- Main use: given language NEW, prove it is not RE by showing $OLD \leq_m NEW$, where OLD known to be not RE.

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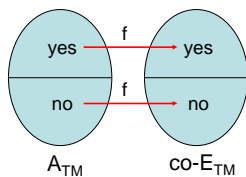
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Many-one reduction example

- Showed E_{TM} undecidable. Consider:

$$co-E_{TM} = \{\langle M \rangle : L(M) \neq \emptyset\}$$



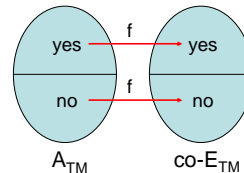
- $f(\langle M, w \rangle) = \langle M' \rangle$ where M' is TM that
 - on input x , if $x \neq w$, then reject
 - else simulate M on x , and accept if M does
- f clearly computable

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Many-one reduction example



- $f(\langle M, w \rangle) = \langle M' \rangle$ where M' is TM that
 - on input x , if $x \neq w$, then reject
 - else simulate M on x , and accept if M does
- f clearly computable
- yes maps to yes?
 - if $\langle M, w \rangle \in A_{TM}$ then $f(\langle M, w \rangle) \in co-E_{TM}$
- no maps to no?
 - if $\langle M, w \rangle \notin A_{TM}$ then $f(\langle M, w \rangle) \notin co-E_{TM}$

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Undecidable problems

Theorem: The language
 $\text{REGULAR} = \{ \langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \}$

is undecidable.

Proof:

- reduce from A_{TM} (i.e. show $A_{\text{TM}} \leq_m \text{REGULAR}$)
- what should $f(\langle M, w \rangle)$ produce?

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Undecidable problems

Proof:

– $f(\langle M, w \rangle) = \langle M' \rangle$ described below

on input x :

- if x has form $0^n 1^n$, accept
- else simulate M on w and accept x if M accepts

• is f computable?

• YES maps to YES?

$\langle M, w \rangle \in A_{\text{TM}} \Rightarrow$
 $f(\langle M, w \rangle) \in \text{REGULAR}$

• NO maps to NO?

$\langle M, w \rangle \notin A_{\text{TM}} \Rightarrow$
 $f(\langle M, w \rangle) \notin \text{REGULAR}$

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Dec. and undec. problems

- the boundary between decidability and undecidability is often quite delicate
 - seemingly related problems
 - one decidable
 - other undecidable
- We will see two examples of this phenomenon next.

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Computation histories

- Recall configuration of a TM: string uqv with $u, v \in \Gamma^*$, $q \in Q$
- The sequence of configurations M goes through on input w is a computation history of M on input w
 - may be accepting, or rejecting
 - reserve the term for halting computations
 - nondeterministic machines may have several computation histories for a given input.

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Linear Bounded Automata

- LBA definition: TM that is prohibited from moving head off right side of input.
- machine prevents such a move, just like a TM prevents a move off left of tape
 - How many possible configurations for a LBA M on input w with $|w| = n$, m states, and $p = |\Gamma|$?
 - counting gives: mnp^n

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Dec. and undec. problems

- two problems we have seen with respect to TMs, now regarding LBAs:
 - LBA acceptance:

$$A_{\text{LBA}} = \{ \langle M, w \rangle : \text{LBA } M \text{ accepts input } w \}$$
 - LBA emptiness:

$$E_{\text{LBA}} = \{ \langle M \rangle : \text{LBA } M \text{ has } L(M) = \emptyset \}$$
- Both decidable? both undecidable? one decidable?

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Dec. and undec. problems

Theorem: A_{LBA} is decidable.

Proof:

- input $\langle M, w \rangle$ where M is a LBA
- key: only mnp^n configurations
- if M hasn't halted after this many steps, it must be looping forever.
- simulate M for mnp^n steps
- if it halts, accept or reject accordingly,
- else reject since it must be looping

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Dec. and undec. problems

Theorem: E_{LBA} is undecidable.

Proof:

- reduce from $co-A_{TM}$ (i.e. show $co-A_{TM} \leq_m E_{LBA}$)
- what should $f(\langle M, w \rangle)$ produce?
- Idea:
 - produce LBA B that accepts exactly the accepting computation histories of M on input w

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Dec. and undec. problems

Proof:

- $f(\langle M, w \rangle) = \langle B \rangle$ described below

on input x , check if x has form

$\#C_1\#C_2\#C_3\#\dots\#C_k\#$

- check that C_1 is the start configuration for M on input w
- check that $C_i \Rightarrow^1 C_{i+1}$
- check that C_k is an accepting configuration for M

• is B an LBA?

• is f computable?

• YES maps to YES?

$\langle M, w \rangle \in co-A_{TM} \Rightarrow$
 $f(\langle M, w \rangle) \in E_{LBA}$

• NO maps to NO?

$\langle M, w \rangle \notin co-A_{TM} \Rightarrow$
 $f(\langle M, w \rangle) \notin E_{LBA}$

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Dec. and undec. problems

- two problems regarding Context-Free Grammars:
 - does a CFG generate all strings:

$$ALL_{CFG} = \{ \langle G \rangle : G \text{ is a CFG and } L(G) = \Sigma^* \}$$
 - CFG emptiness:

$$E_{CFG} = \{ \langle G \rangle : G \text{ is a CFG and } L(G) = \emptyset \}$$
- Both decidable? both undecidable? one decidable?

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Dec. and undec. problems

Theorem: E_{CFG} is decidable.

Proof:

- observation: for each nonterminal A , the set

$S_A = \{ w : A \Rightarrow^* w \}$

is non-empty iff there is some rule:

$A \rightarrow x$

and \forall non-terminals B in string x , $S_B \neq \emptyset$

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Dec. and undec. problems

Proof:

- on input $\langle G \rangle$
- mark all terminals in G
- repeat until no new non-terminals get marked:
 - if there is a production $A \rightarrow x_1 x_2 x_3 \dots x_k$
 - and each symbol x_1, x_2, \dots, x_k has been marked
 - then mark A
- if S has been marked, accept, else reject.
- terminates? correct?

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