CS 153 Current Topics in Theoretical Computer Science **Spring 2016**

Problem Set 1

Out: April 19 *Due:* **May 3**

You are encouraged to work in groups of two or three; however you must turn in your own write-up and note with whom you worked. You may consult the course notes and resources linked from the webpage. Any other general informational resources (such as Wikipedia pages) is fine but please do not seek out solutions to the specific problems.

1. Prove that $R(\langle n,1,k\rangle \oplus \langle 1,m,1\rangle) = nk+m$.
2. The *asymptotic rank* is denoted $\tilde{R}$ and defined on tensors T as follows:
$$\tilde{R}(T) = \lim_{n\to\infty} R(T^{\otimes n})^{1/n}.$$
Give an example of a tensor T for which $\tilde{R}(T) < R(T)$.
3. Given two univariate polynomials of degree at most n,
$$a(X) = \sum_{i=0}^{n} a_i X^i \qquad \text{and} \qquad b(X) = \sum_{i=0}^{n} b_i X^i,$$
their product is a polynomial of degree at most $2n$. Write down the $2n+1$ bilinear forms in the a_i and b_i variables that are computed in the product. Show that the rank of the associated tensor is at most $2n+1$ by reduction to group algebra multiplication over a group G you specify.
4. This problem concerns the rank of $C[G]$-multiplication. Denote by T_G the tensor associated with $C[G]$-multiplication. We will also need to work with families of groups $\{G_i\}$. For example one might have G_i equal to the symmetric group S_i.
   (a) Show that if $\omega = 2$, then $R(T_{G_i}) \le O(|G_i|^{1+\epsilon})$ for all $\epsilon > 0$.
   (b) Let $\{G_i\}$ be a family of groups with $d_{\max}(G_i) \ge f(i)$ for $f(i)$ a function that grows faster than a constant. Show that $R(T_{G_i})) \le O(|G_i|^{1+\epsilon})$ for all $\epsilon > 0$ implies $\omega = 2$.
   (c) Show that $R(T_{G_i}) \le O(|G_i|^{3/2})$ unconditionally.
5. Show that $R(T_{S_3}) \le 9$.
6. Suppose that $X, Y, Z \subseteq G$ satisfy the Triple Product Property, and one of X, Y, Z is a normal subgroup. Show that $|X||Y||Z| \le |G|$.
7. Show that if G is non-abelian, then there exists $X, Y, Z \subseteq G^3$ that satisfy the Triple Product Property and for which $|X||Y||Z| > |G^3|$. Hint: G is abelian iff all *commutators* $[x, y] = xyx^{-1}y^{-1}$ equal 1. Your construction will produce two subgroups of size G and one subset of size $|G| + 1$.