CS151 Complexity Theory

Lecture 18
June 1, 2023

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Natural Proofs

- Razborov and Rudich defined the following "natural" format for circuit lower bounds:
  - identify property **P** of functions $f:\{0,1\}^* \rightarrow \{0,1\}$
  - **P** $= \cup_n$ $\mathbf{P}_n$ is a natural property if:
    - (useful) $\forall n$ $f_n \in \mathbf{P}_n$ implies f does not have poly-size circuits [i.e. $f_n \in \mathbf{P}_n$ implies ckt size $\geq s(n) >> poly(n)$]
    - (constructive) can decide "$f_n \in \mathbf{P}_n$?" in poly time given the *truth table* of f_n
    - (large) at least $(½)^{O(n)}$ fraction of all 2^{2^n} functions on n bits are in $\mathbf{P}_n$
  - show some function family $g = \{g_n\}$ is in $\mathbf{P}_n$

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Natural Proofs

- *all known circuit lower bound techniques* are natural for a suitably parameterized version of the definition

Theorem (RR): if there is a 2^{n^ϵ}-OWF, then there is no natural property **P**.
  - factoring believed to be 2^{n^ϵ}-OWF
  - general version also rules out natural properties useful for proving many other separations, under similar cryptographic assumptions

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Natural Proofs

- Proof:
  - **main idea**: natural property $\mathbf{P}_n$ can efficiently distinguish

    pseudorandom functions

    from

    truly random functions
  - but cryptographic assumption implies existence of *pseudorandom functions* for which this is impossible

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Proof (continued)

- Recall: assuming One-Way-Permutations

  $f_k:\{0,1\}^k \rightarrow \{0,1\}^k$

  that are not invertible by 2^{k^ϵ} size circuits

- we constructed PRG $G:\{0,1\}^k \rightarrow \{0,1\}^{2k}$
  - no circuit C of size $s = 2^{k^\delta}$ for which

    $|Pr_x[C(G(x)) = 1] - Pr_z[C(z) = 1]| > 1/s$

(BMY construction with slightly modified parameters)

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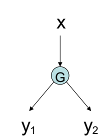
Proof (continued)

- Think of G as $G:\{0,1\}^k \rightarrow \{0,1\}^k \times \{0,1\}^k$

  $G(x) = (y_1, y_2)$

- Graphically:



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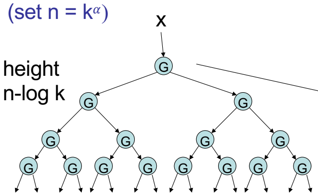
Proof (continued)

- A function $F:\{0,1\}^k \to \{0,1\}^{2^n}$
  (set $n = k^{\alpha}$)

height n-log k



Given x, i, can compute i-th output bit in time n·poly(k)

each x, defines a poly-time computable function f_x

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Proof (continued)

(useful) $\forall n\ f_n \in \mathbf{P}_n \Rightarrow f$ does not have poly-size circuits
(constructive) "$f_n \in \mathbf{P}_n$?" in poly time given *truth table* of f_n
(large) at least $(½)^{O(n)}$ fraction of all 2^{2^n} fns. on n-bits in $\mathbf{P}_n$

- f_x in poly-time $\Rightarrow$ for all x: $f_x \notin \mathbf{P}_n$ (useful)
- $\Pr_g[g \in \mathbf{P}_n] \geq (1/2)^{O(n)}$ (large)
- constructive: exists circuit $T:\{0,1\}^{2^n} \to \{0,1\}$ of size $2^{O(n)}$ for which
  $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

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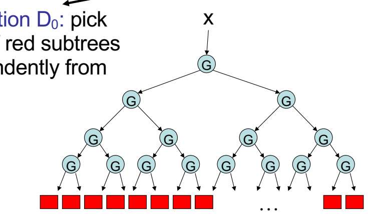
Proof (continued)

- $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_0: pick roots of red subtrees independently from $\{0,1\}^k$



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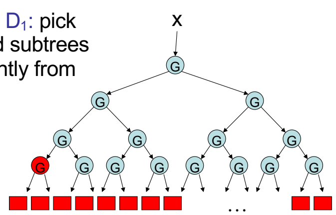
Proof (continued)

- $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_1: pick roots of red subtrees independently from $\{0,1\}^k$



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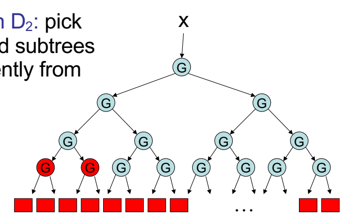
Proof (continued)

- $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_2: pick roots of red subtrees independently from $\{0,1\}^k$



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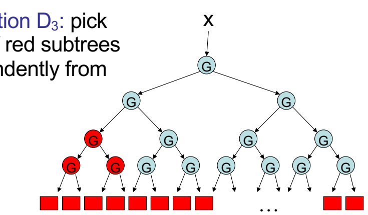
Proof (continued)

- $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_3: pick roots of red subtrees independently from $\{0,1\}^k$



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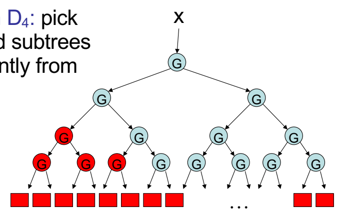
Proof (continued)

- $|Pr_x[T(f_x) = 1] - Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_4: pick roots of red subtrees independently from $\{0,1\}^k$



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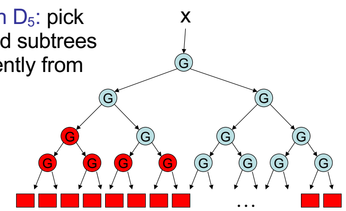
Proof (continued)

- $|Pr_x[T(f_x) = 1] - Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_5: pick roots of red subtrees independently from $\{0,1\}^k$



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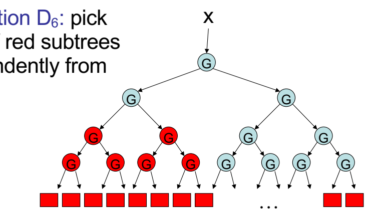
Proof (continued)

- $|Pr_x[T(f_x) = 1] - Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_6: pick roots of red subtrees independently from $\{0,1\}^k$



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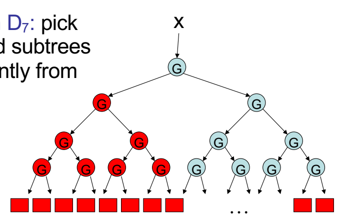
Proof (continued)

- $|Pr_x[T(f_x) = 1] - Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution D_7: pick roots of red subtrees independently from $\{0,1\}^k$



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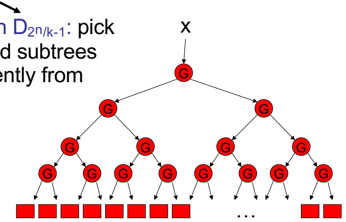
Proof (continued)

- $|Pr_x[T(f_x) = 1] - Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_{2n/k-1}$: pick roots of red subtrees independently from $\{0,1\}^k$



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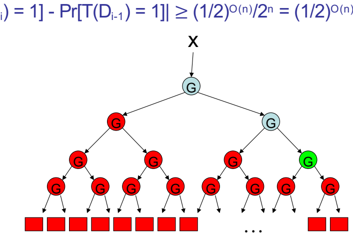
Proof (continued)

– For some i:
$|Pr[T(D_i) = 1] - Pr[T(D_{i-1}) = 1]| \geq (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$



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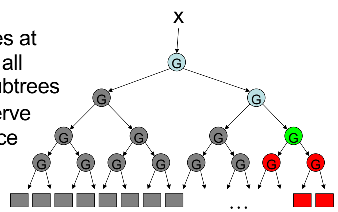
Proof (continued)

– For some i:

$|\Pr[T(D_i) = 1] - \Pr[T(D_{i-1}) = 1]| \ge (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$

fix values at roots of all other subtrees to preserve difference



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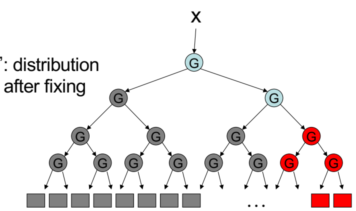
Proof (continued)

– For some i:

$|\Pr[T(D_i') = 1] - \Pr[T(D_{i-1}') = 1]| \ge (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$

D_i': distribution D_i after fixing



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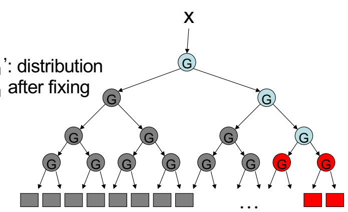
Proof (continued)

– For some i:

$|\Pr[T(D_i') = 1] - \Pr[T(D_{i-1}') = 1]| \ge (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$

D_{i-1}': distribution D_{i-1} after fixing



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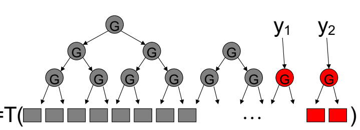
Proof (continued)

$|\Pr[T(D_i') = 1] - \Pr[T(D_{i-1}') = 1]| \ge (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$



– $C(y_1,y_2)=T($ ... $)$

$|\Pr_x[C(G(x)) = 1] - \Pr_{y_1, y_2}[C(y_1, y_2) = 1]| \ge (1/2)^{O(n)}$

T(D_i') T(D_{i-1}')

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Proof (continued)

– recall: no circuit C of size $s = 2^{k^\delta}$ for which:

$|\Pr_x[C(G(x)) = 1] – \Pr_{y_1, y_2}[C(y_1, y_2) = 1]| > 1/s$

– we have C of size $2^{O(n)}$ for which:

$|\Pr_x[C(G(x)) = 1] - \Pr_{y_1, y_2}[C(y_1, y_2) = 1]| \ge (1/2)^{O(n)}$

– with $n = k^\alpha$, α arbitrary constant

– set α such that $2^{O(n)} < s$

– contradiction.

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Natural Proofs

- To prove circuit lower bounds, we must either:
  - Violate largeness: seize upon an incredibly specific feature of hard functions (one not possessed by a random function !)
  - Violate constructivity: identify a feature of hard functions that cannot be computed efficiently from the **truth table**
- no “non-natural property” known for all but the very weakest models…

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"We do not conclude that researchers should give up on proving serious lower bounds.

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"We do not conclude that researchers should give up on proving serious lower bounds. Quite the contrary, by classifying a large number of techniques that are unable to do the job, we hope to focus research in a more fruitful direction. Pessimism will only be warranted if a *long period of time* passes without the discovery of a non-naturalizing lower bound proof."

Rudich and Razborov
1994

Moral

- To resolve central questions:
  - avoid relativizing arguments
    - use PCP theorem and related results
    - focus on circuits, etc…
  - avoid constructive arguments
  - avoid arguments that yield lower bounds for random functions

Course Summary

Course summary

- Time and space
  - hierarchy theorems
  - FVAL in **L**
  - CVAL **P**-complete
  - QSAT **PSPACE**-complete
  - succinct CVAL **EXP**-complete

Course summary

- Non-determinism
  - NTIME hierarchy theorem
  - "NP-intermediate" problems (Ladner's Theorem)
  - unary languages (likely) not NP-complete
  - Savitch's Theorem
  - Immerman-Szelepcsényi Theorem

  Problem sets:
  - *sparse* languages (likely) not NP-complete

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Course summary

- Non-uniformity
  - formula lower bound (Andreev, Hastad)
  - monotone circuit lower bound (Razborov)

  Problem sets:
  - Barrington's Theorem
  - formula lower bound for parity

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Course summary

- Randomness
  - polynomial identity testing + Schwartz-Zippel
  - unique-SAT (Valiant-Vazirani Theorem)
  - Blum-Micali-Yao PRG
  - Nisan-Wigderson PRG
  - worst-case hardness ⇒ average-case hardness
  - Trevisan extractor

  Problem sets:
  - Goldreich-Levin hard bit

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Course summary

- Alternation
  - $QSAT_i$ complete for levels of the **PH**
  - Karp-Lipton theorem
  - **BPP** in **PH**

  Problem sets:
  - approximate counting + sampling with an **NP**-oracle
  - **VC**-dimension is Σ_3-complete
  - the class S_2^P (final)

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Course summary

- Counting
  - #matching is **#P**-complete

  Problem sets:
  - permanent is **#P**-complete
  - Toda's theorem: $\mathbf{PH} \subseteq \mathbf{P}^{\#\mathbf{P}}$

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Course summary

- Interaction
  - **IP = PSPACE**
  - GI in **NP** ∩ **coAM**
  - using NW PRG for **MA**, variant for **AM**
  - hardness of approximation , PCPs
  - elements of the PCP theorem

  Problem sets:
  - BLR linearity test
  - Clique hard to approximate to within N^{ϵ}

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Course summary

- Barriers to progress
  - oracles rule out relativizing proofs
  - “natural proofs” rule out many circuit lower bound techniques

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Course summary

• Time and space	**L, P, PSPACE, EXP**
• Non-determinism	**NL, NP, coNP, NEXP**
• Non-uniformity	**NC, P/poly**
• Randomness	**RL, ZPP, RP, coRP, BPP**
• Alternation	**PH, PSPACE**
• Counting	**#P**
• Interaction	**IP, MA, AM, PCP[log n, 1]**

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The big picture

- All classes on previous slide are probably distinct, except:
  - **P, ZPP, RP, coRP, BPP** (probably all equal)
  - **L, RL** (probably all equal; **NL**?)
  - **NP, MA, AM** (probably all equal)
  - **IP = PSPACE**
  - **PCP[log n, 1] = NP**
- Only real separations we know separate classes delimiting same resource:
  - e.g. **L ≠ PSPACE, NP ≠ NEXP**

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The big picture

Remember:

possible explanation for failure to prove conjectured separations…

…is that they are false

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The big picture

- Important techniques/ideas:
  - simulation and diagonalization
  - reductions and completeness
  - self-reducibility
  - encoding information using low-degree polynomials
  - randomness
  - others…

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The big picture

- I hope you take away:
  - an ability to extract the essential features of a problem that make it hard/easy…
  - knowledge and tools to connect computational problems you encounter with larger questions in complexity
  - background needed to understand current research in this area

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The big picture

- background to *contribute* to current research in this area
  - many open problems
  - young field
  - try your hand…