

Suggested references are listed on the course web page. You're welcome to take advantage of other references. However, realize that some of the problems I'll assign are "standards," whose solutions might be found in textbooks, notebooks of older students, and so forth. Regardless of the source, don't read existing solutions before turning in your work. This includes materials from previous offerings of this course.

Collaboration: Unless a problem is marked "no collaboration," you can work with other students in the class (I encourage it). But you should produce the final write-up for submission alone, after all discussions, and if you worked with someone on a problem, you should list their name by your solution. You can discuss "no collaboration" problems with the TA or with me. I won't usually discuss material over email.

The TA is Kevin Ko. Please leave assignments in his mailbox (in Jorgensen) on the due date.

1. [No collaboration] Mitzenmacher & Upfal 2.20.
2. M&U 2.26.
3. M&U 2.32. (This is a famous problem known as the "secretary problem.")
4. Show that the  $d$ -dimensional Gaussian densities (parameterized by  $\sigma$ ),  $(\sqrt{2\pi}\sigma)^{-d}e^{-(2/\sigma^2)\sum x_i^2}$ , are the only continuously differentiable probability densities in  $\mathbb{R}^d$  that are spherically symmetric (invariant under orthogonal transformations) and whose coordinates (the  $x_i$ ) are statistically independent.
5. [No collaboration] Here's a nice example of the *probabilistic method* in combinatorics. There are  $n^2$  light bulbs arranged in a square grid. Initially some are lit and some are not. The initial assignment is specified with variables  $\{a_{ij}\}_{i,j \in \{1, \dots, n\}}$ , where  $a_{ij} = 1$  if the bulb in row  $i$  and column  $j$  is on, and  $a_{ij} = -1$  if the bulb is off. Now, in each row and each column there is a switch which can simultaneously flip all the bulbs in that row or column. Let  $x_i$  denote the switch for row  $i$ , where  $x_i = 1$  means that the row is left alone, and  $x_i = -1$  means that it is flipped. Similarly let  $y_j \in \pm 1$  denote the switch for column  $j$ .

We're interested in the following function:  $F(n) = \min_{a_{ij} \in \pm 1} \max_{x_i, y_j \in \pm 1} \sum_{i,j} a_{ij}x_iy_j$ . In other words, how can the initial assignment be set so as to minimize the maximum number of bulbs that can be lit?

Show that  $F(n) \leq (2 \log^{1/2} 2)n^{3/2}$ . (Natural log.)

6. Let  $G = (V, E)$  be an undirected graph without loops or multiple edges, and let  $V$  be the disjoint union of  $V_B$ , the "boundary" vertices, and  $V_I$ , the "interior" vertices. Let  $u \sim v$  signify that vertices  $u$  and  $v$  are neighbors. Let  $\mathbb{C}^V$  be the set of complex-valued functions on  $V$ . The *Laplacian operator* is the mapping

$$L : \mathbb{C}^V \rightarrow \mathbb{C}^V$$
$$(Lf)(v) = \sum_{u \sim v} (f(v) - f(u))$$

We are interested in the special case that  $V = \{0, 1, 2, \dots\}$ ,  $E = \{(i, i+1) : i \geq 0\}$ ,  $V_B = \{0\}$ ,  $V_I = \{1, 2, \dots\}$ . Show: (a) If  $(Lf)(v) = 0 \forall v \in V_I$  then  $f$  is a linear function. (b) If for some  $c$ ,  $(Lf)(v) = c \forall v \in V_I$  then  $f$  is a quadratic function.

Now consider symmetric random walk on the integers. (c) Show that for any integers  $v, w$ , a walk starting at  $v$  has probability 1 of eventually hitting  $w$ . (d) Show that the expected time until the walk hits  $w$  is infinite.