

Outline

- Nondeterminism
- Regular expressions
- Elementary reductions

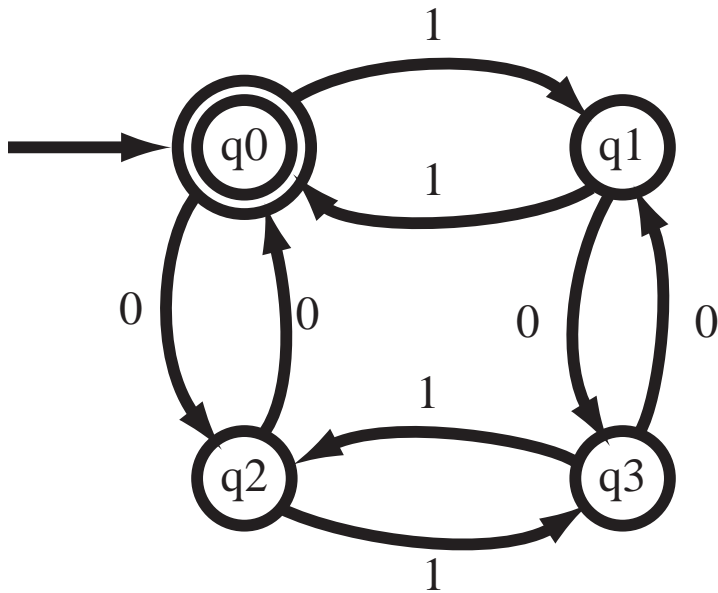


Deterministic Finite Automata

- A *finite automaton* is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$
 - Q is a *finite* set of *states*
 - Σ is an alphabet
 - $\delta : Q \times \Sigma \rightarrow Q$ is a *transition function*
 - $q_0 \in Q$ is the *initial state*
 - $F \subseteq Q$ is a set of *final* or *accepting* states



Transition diagrams



$$Q = \{q_0, q_1, q_2, q_3\}$$

$$\Sigma = \{0, 1\}$$

$$\delta =$$

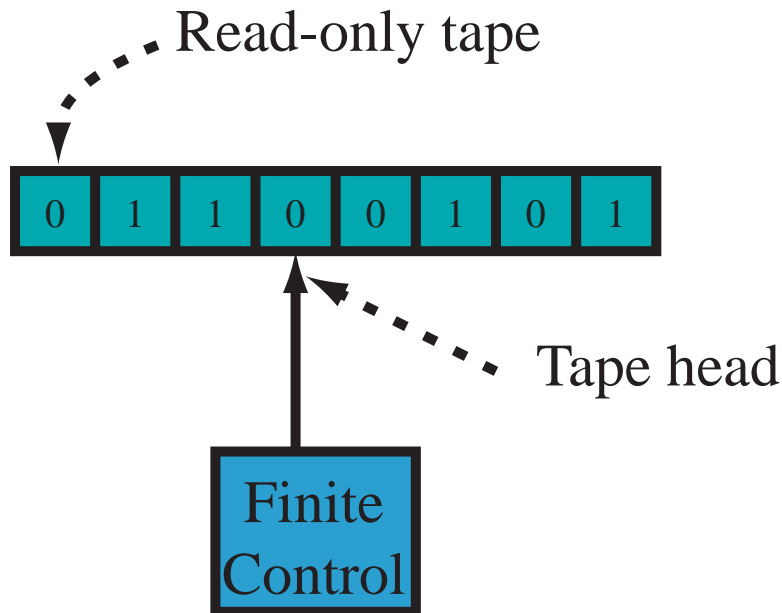
State	Symbol	Goto
q_0	0	q_2
q_0	1	q_1
q_1	0	q_3
q_1	1	q_0
q_2	0	q_0
q_2	1	q_3
q_3	0	q_1
q_3	1	q_2

$$q_0 = q_0$$

$$F = \{q_0\}$$



A finite automaton



1. In state q
 - a. read a symbol c
 - b. move the tape head right
 - b. goto state $\delta(q, c)$
2. Accept iff the FA is in a final state after reading the last symbol



Definition of regular expressions

- Let Σ be an alphabet. The regular expressions are defined inductively as follows:
 - \emptyset is a regular expression denoting $\{\}$,
 - ϵ is a regular expression denoting $\{\epsilon\}$,
 - for each $a \in \Sigma$, a is a regular expression denoting $\{a\}$,
 - Assume r and s are regular expressions denoting sets R and S , then
 - * rs denotes RS ,
 - * $r + s$ denoted $R \cup S$,
 - * r^* denotes R^*



Examples of regular expressions

- abc denotes $\{abc\}$,
- a^*b denotes $\{b, ab, aab, aaab, \dots\}$,
- $(a + b)^*aa(a + b)^*$: all strings containing at least two consecutive a 's.
- $(0^*(101^*01)^*)^*$: all binary numbers that are a multiple of 3.



Nondeterminism

- For *deterministic* finite automata, the transition function **delta** is a *function*
 - *delta : state * symbol -> state*
 - *For each state, and each symbol, there is a unique next state*
- What if we relax this?
- In a nondeterministic finite automaton, **delta** is a transition *relation*



Definition of an NFA

- A *nondeterministic finite automaton* is a 5-tuple $(Q, \Sigma, \delta, s, F)$
 - Q is a *finite* set of *states*
 - Σ is an *alphabet*
 - $\delta : Q \times \Sigma \rightarrow 2^Q$ is a *transition relation*
 - $s \in Q$ is the *initial state*
 - $F \subseteq Q$ is a set of *final* or *accepting* states

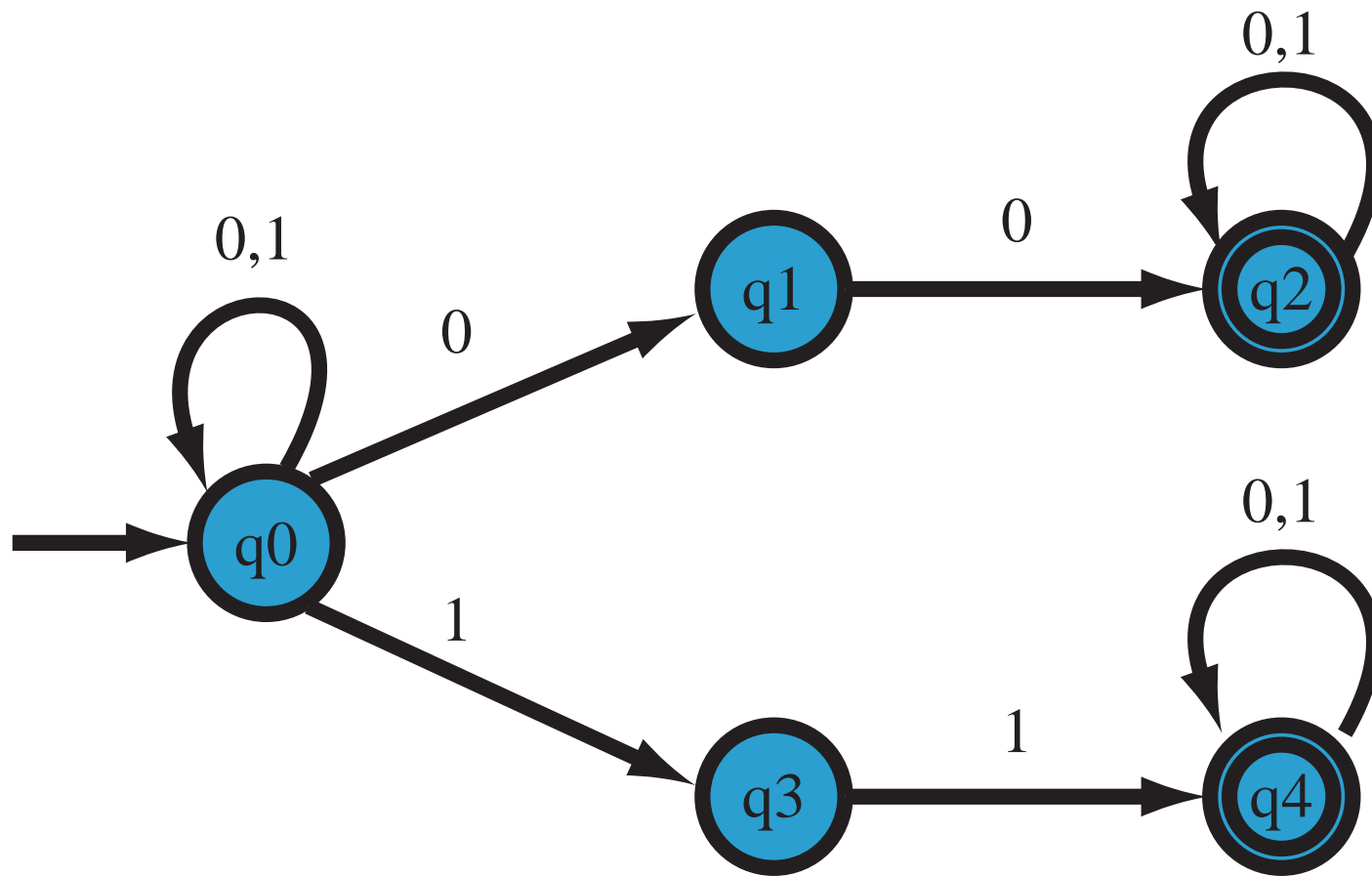


The delta relation

- Note, the following definitions are equivalent:
 - $\delta : Q \times \Sigma \rightarrow 2^Q$ is a *transition relation*
 - $\delta \subseteq Q \times \Sigma \times Q$ is a *transition relation*
- The transition diagram is often drawn without empty edges:
 - if $\delta(q, c) = \{\}$, then the machine rejects and we do not draw the edge



Example NFA



Executions (for DFAs)

- An *execution* of a **deterministic** automaton $M = (Q, \Sigma, \delta, q_0, F)$ is a sequence

$$(q_0, c_0)(q_1, c_1) \dots (q_{n-1}, c_{n-1})q_n \in (Q \times \Sigma)^*$$

such that:

- q_0 is the start state
 - $\delta(q_i, c_i) = q_{i+1}$
- The *execution string* s is $c_0c_1 \dots c_{n-1}$
 - If there exists an execution for string s with $q_n \in F$ the automaton *accepts* and $s \in L(M)$; otherwise it *rejects*



Executions (for NFAs)

- An *execution* of a **nondeterministic** automaton $M = (Q, \Sigma, \delta, q_0, F)$ is a sequence

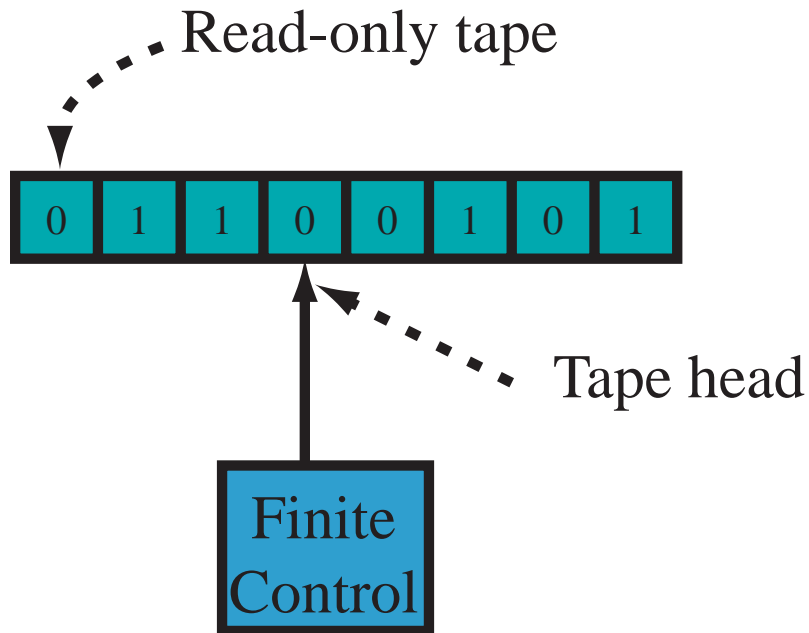
$$(q_0, c_0)(q_1, c_1) \dots (q_{n-1}, c_{n-1})q_n \in (Q \times \Sigma)^*$$

such that:

- q_0 is the start state
 - $q_{i+1} \in \delta(q_i, c_i)$
- The *execution string* s is $c_0c_1 \dots c_{n-1}$
 - If there exists an execution for string s with $q_n \in F$ the automaton *accepts* and $s \in L(M)$; otherwise it *rejects*



A nondeterministic finite automaton



1. In state q
 - a. read a symbol c
 - b. move the tape head right
 - b. goto some state in $\delta(q, c)$
2. Accept iff the NFA is in a final state after reading the last symbol



Execution models

1. Classical: For each state q and input symbol c , the automaton makes a nontermistic choice among all the possible options $\delta(q, c)$; it accepts if it made all the right choices
2. Another choice:
 - At any point, the automaton is in a *superposition* of states \mathcal{Q}_1
 - For each symbol c , it enters states \mathcal{Q}_2 where $q_2 \in \mathcal{Q}_2$ iff $\exists q_1 \in \mathcal{Q}_1. q_2 \in \delta(q_1, c)$



Extending the transition relation

- Defining $\hat{\delta}$:
 - $\hat{\delta}(q, \epsilon) = \{q\}$
 - $\hat{\delta}(q, wa) = \{p \mid \exists r \in \hat{\delta}(q, w). p \in \delta(r, a)\}$
- A machine $M = (Q, \Sigma, \delta, q_0, F)$ accepts string s iff $\hat{\delta}(q_0, s) \cap F \neq \{\}$



Expressive power of NFAs vs. DFAs

- Every DFA is an NFA
- NFAs allow more transitions
- Is the language more expressive?
 - *Is there a language $L=L(M)$ for some NFA M , where $L \neq L(M')$ for some DFA M' ?*



NFA/DFA Equivalence

- To prove equivalence of DFAs and NFAs we must do two things:
 - I: For each DFA, produce an NFA that accepts the same language
 - II: For each NFA, produce a DFA that accepts the same language
- This is called a *reduction*—reduce one construction to another equivalent one
 - We'll do part II; part I is immediate



NFA \rightarrow DFA construction

- Consider an arbitrary NFA $M = (Q, \Sigma, \delta, s, F)$; construct an equivalent DFA $M' = (Q', \Sigma, \delta', s', F')$
 - Let $Q' = 2^Q$
 - Let $s' = \{s\}$
 - Let $\delta'(\{q_1, \dots, q_n\}, c) = \bigcup_{i=1}^n \delta(q_i, c)$
 - Let $F' = \{t \in 2^Q \mid t \cap F \neq \{\}\}$



NFA- \rightarrow DFA construction: verification

Theorem $\delta(s, x) = \delta'(\{s\}, x)$

Corollary $\delta(s, x) \cap F \neq \{\} \Leftrightarrow \delta'(\{s\}, x) \cap F' \neq \{\}$



NFA- \rightarrow DFA construction: base case

Theorem $\delta(s, x) = \delta'(\{s\}, x)$

Base case $\delta(s, \epsilon) = \{s\} = \delta'(\{s\}, \epsilon)$



NFA \rightarrow DFA construction: induction step

- Assume $\delta(s, x) = \delta'(\{s\}, x)$ for some x
- Prove $\delta(s, xa) = \delta'(\{s\}, xa)$ for any a

$$\begin{aligned} & \delta(s, xa) \\ = & \text{by definition of } \hat{\delta} \\ & \delta(\delta(s, x), a) \\ = & \text{by definition of } \hat{\delta} \\ & \bigcup_{p \in \delta(s, x)} \delta(p, a) \\ = & \text{by induction} \\ & \bigcup_{p \in \delta'(s, x)} \delta(p, a) \\ = & \text{by definition of } \delta' \\ & \bigcup_{p \in \delta'(s, x)} \delta'(\{p\}, a) \\ = & \text{by definition of } \delta' \\ & \bigcup_{p \in \delta'(\{s\}, x)} \delta'(\{p\}, a) \\ = & \text{by definition of } \delta' \\ & \delta'(\delta'(\{s\}, x), a) \\ = & \text{by definition of } \delta' \\ & \delta'(\{s\}, xa) \end{aligned}$$

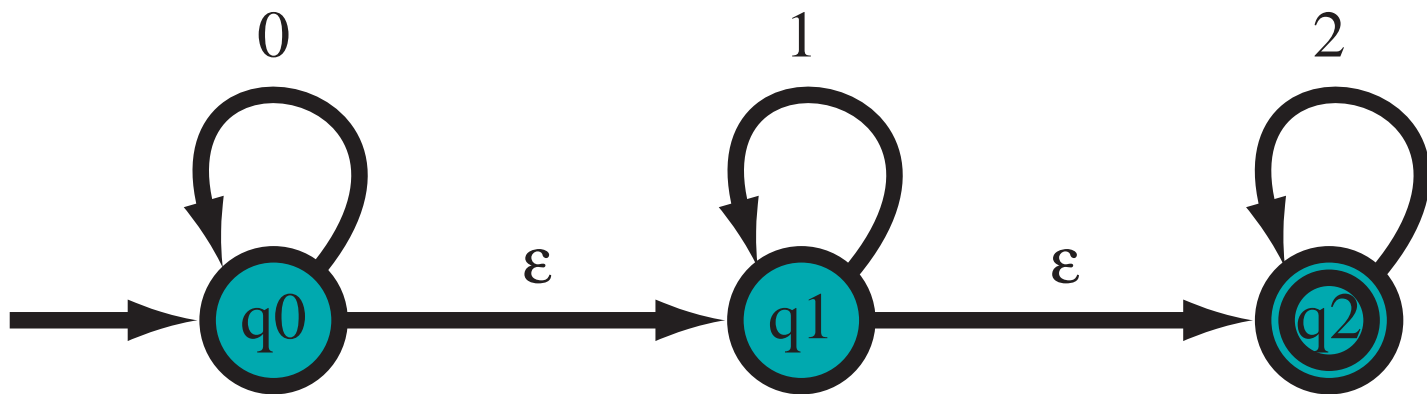


NFA with epsilon transitions

- Suppose we add a new twist
 - *Allow transitions that require no input*
 - *Label them epsilon (no input)*
 - *The automaton may choose to take an epsilon transition without any input*
- Executions include $(q_i, \text{epsilon}) (q_{i+1}, _)$ steps
- A machine accepts if there exists an execution that accepts



Example e-NFA



E-NFA definition

- A *nondeterministic finite automaton with ϵ transitions* is a 5-tuple $(Q, \Sigma, \delta, s, F)$
 - Q is a *finite* set of *states*
 - Σ is an alphabet
 - $\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^Q$ is a *transition relation*
 - $s \in Q$ is the *initial state*
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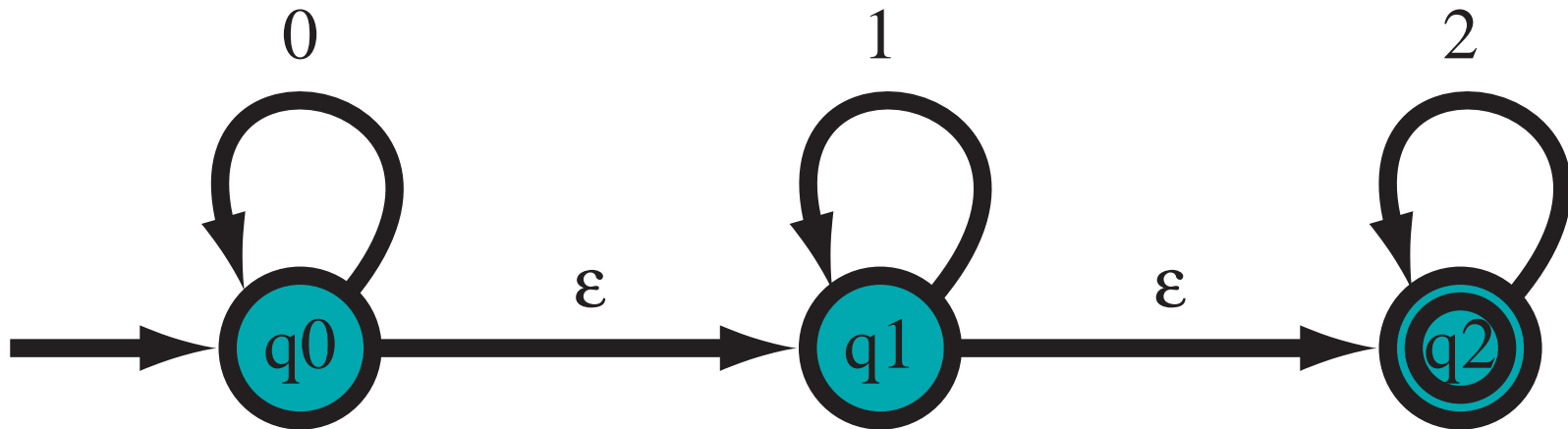


E-NFA- \rightarrow NFA reduction

- Consider the superposition model
 - *If the e-NFA could be in states $\{q_1, q_2, \dots, q_n\}$, add the states that could be reached by e-transitions from any state q_i*
 - *This is called the e-closure*



E-closure example



$$\epsilon\text{-closure}(q_0) = \{ q_0, q_1, q_2 \}$$

$$\epsilon\text{-closure}(q_1) = \{ q_1, q_2 \}$$

$$\epsilon\text{-closure}(q_2) = \{ q_2 \}$$



Formal definition

- Define ϵ -closure(P) = $\bigcup_{p \in P} \epsilon$ -closure(p)
 - Let $\hat{\delta}(q, \epsilon) = \epsilon$ -closure(q)
 - Let $\hat{\delta}(q, xa) = \epsilon$ -closure(P), where $P = \{p \mid \exists r \in \hat{\delta}(q, w). p \in \delta(r, a)\}$
 - Define $\delta(R, a) = \bigcup_{r \in R} \delta(r, a)$
 - Define $\hat{\delta}(R, a) = \bigcup_{r \in R} \hat{\delta}(r, a)$
- $M = (Q, \Sigma, \delta, s, F)$ accepts string x iff $\hat{\delta}(s, x) \cap F \neq \{\}$



Reduction

- Let $M = (Q, \Sigma, \delta, s, F)$ be an NFA with ϵ moves.
- Construct $M' = (Q, \Sigma, \delta', s, F')$ an NFA without ϵ moves.
 - Let $\delta'(q, a) = \hat{\delta}(q, a)$
 - Let $F' = \begin{cases} F \cup \{s\} & \text{if } \epsilon\text{-closure}(s) \cap F \neq \{\} \\ F & \text{otherwise} \end{cases}$



Proof

Prove $\delta'(s, x) = \hat{\delta}(s, x)$ by induction on the length of x

Note that $\delta'(s, \epsilon) = \hat{\delta}(s, \epsilon)$ may not be true. Oh well, start the induction at $|x| = 1$.

Base If $|x| = 1$ then $x = a$ and $\delta'(s, a) = \hat{\delta}(s, a)$ by definition



Induction step

Step Suppose $x = wa$

$$\begin{aligned} & \delta'(s, wa) \\ = & \text{by definition} \\ & \delta'(\delta'(s, w), a) \\ = & \text{by induction} \\ & \delta'(\hat{\delta}(s, w), a) \\ = & \text{by definition} \\ & \bigcup_{q \in \hat{\delta}(s, w)} \delta'(q, a) \\ = & \text{by definition of } \delta' \\ & \bigcup_{q \in \hat{\delta}(s, w)} \hat{\delta}(q, a) \\ = & \text{by definition of } \hat{\delta} \\ & \hat{\delta}(s, wa) \end{aligned}$$



Summary (so far)

- Proofs so far
 - *Proved NFA- \rightarrow DFA*
 - *Proved e-NFA- \rightarrow NFA*
- Next, establish total equivalence
 - *Prove regex- \rightarrow e-NFA*
 - *Prove DFA- \rightarrow regex*
- If so, all four are equivalent



Definition of regular expressions

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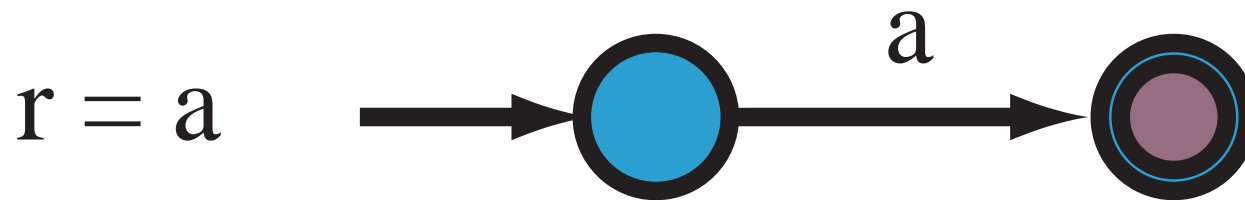


Regex \rightarrow eNFA

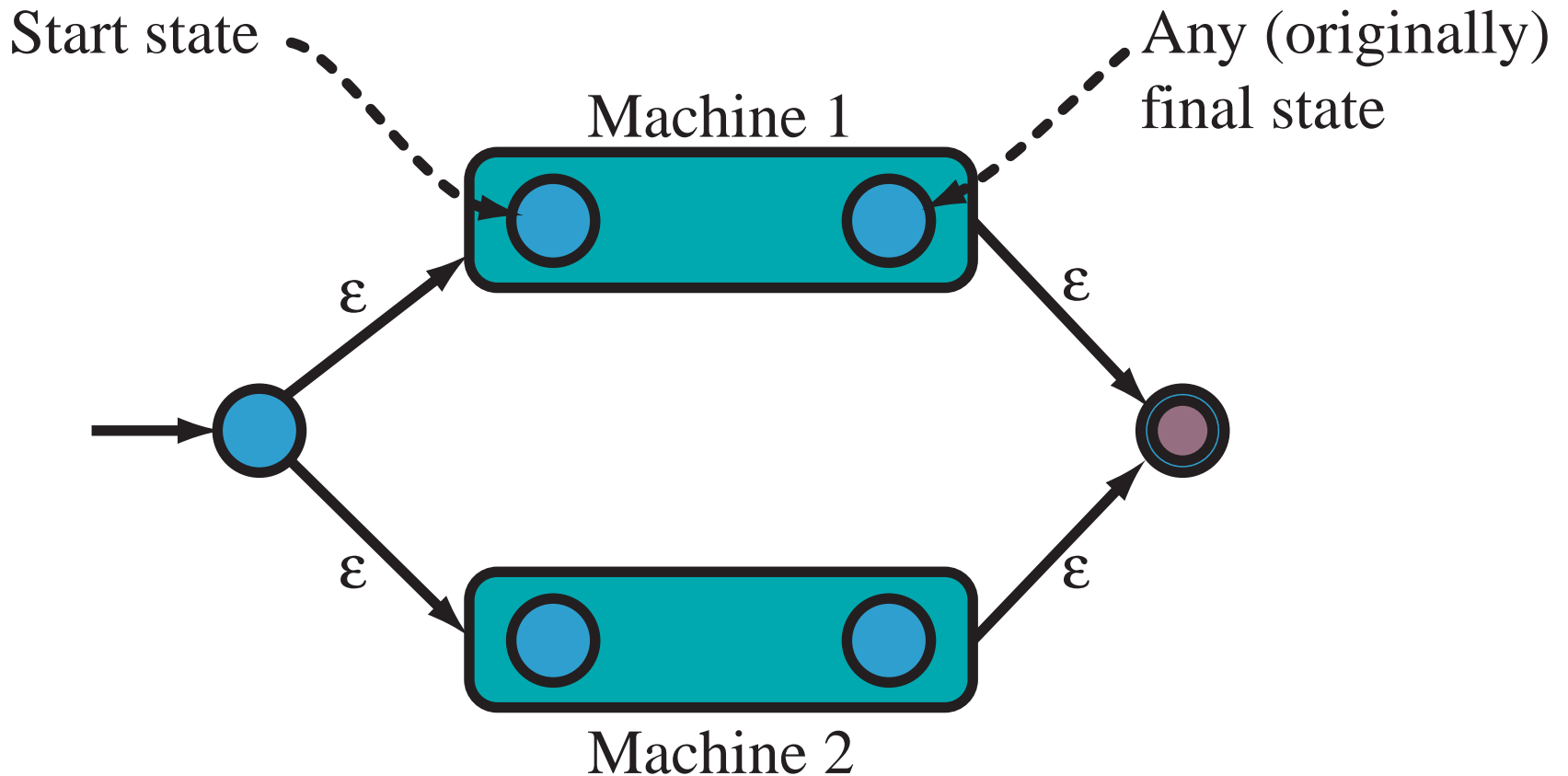
- Prove by *structural induction*: induction on the size of the regular expression
- Base cases:
 - Show the empty RE has an e-NFA
 - Show the ϵ RE has an e-NFA
 - Show the a (symbol) RE has an e-NFA
- Induction:
 - Show (xy) has an e-NFA
 - Show $(x + y)$ has an e-NFA
 - Show x^* has an e-NFA



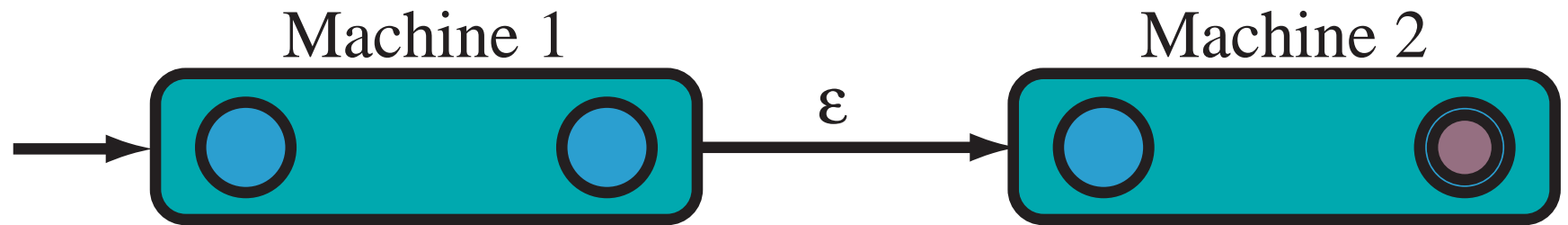
Base cases



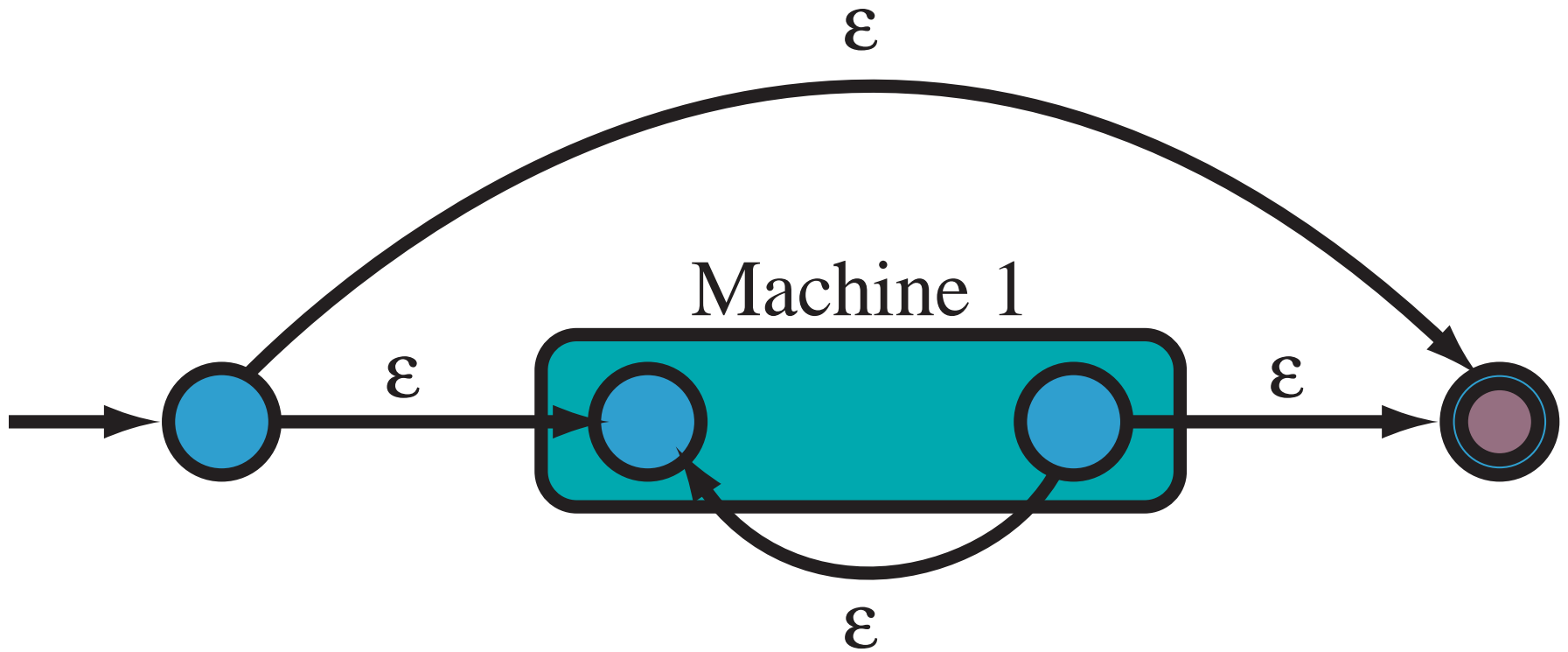
Choice



Concatenation



Kleene closure



Final proof: DFA- \rightarrow regex

- Think about it

