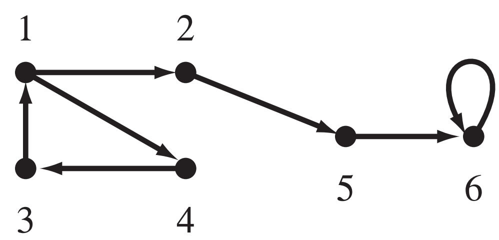
Exercise 1. Undirected graphs

Show that if an undirected graph has more than $(n-1)(n-2)/2$ edges, then it must be connected.

Exercise 2. Directed graphs

Consider a directed graph $G = (V, E)$, where $|V| = n$ and $|E| = m$. An *adjacency matrix* for the graph is an $n \times n$ matrix with entries that are either 0 or 1, where entry $e_{ij} = 1$ iff $(v_i, v_j) \in E$. For example, the figure below shows a graph and its adjacency matrix.



Directed graph

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	1	0	0	0	0	0
4	0	0	1	0	0	0
5	0	0	0	0	0	1
6	0	0	0	0	0	1

Adjacency matrix

The *transitive closure* of a directed graph $G = (V, E)$ is a directed graph $T = (V, E_2)$ such that if there is a path from u to v in G, then $(u, v) \in E_2$.

a) Describe an algorithm to construct the adjacency matrix for T from the adjacency matrix of G.

b) **Hard** Under the assumption that it takes time $M(n)$ to multiply two $n \times n$ Boolean matrices, give an algorithm that takes time $O(M(n))$.

Exercise 3. NP-complete problems

The *integer programming* problem is defined as follows. Given rational numbers a_{ij}, c_j, and b_i for $1 \le i \le m$ and $1 \le j \le n$, find integers $x_1, x_2, \ldots, x_n$ that maximize the linear function

$$\sum_{j=1}^{n} c_j x_j$$

subject to the linear constraints

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (1 \le i \le m).$$

Assume that integer solutions, if they exist, require only polynomially many bits in the size of the problem. Show that integer programming is NP-complete.

Exercise 4. Cook's theorem

Reduce the k-coloring problem directly to the CNF-SAT problem.