

Exercise 1. Context-free languages

Which of the following are CFLs?

a $\{a^i b^j \mid i \neq j \wedge i \neq 2j\}$

This is a CFL. Consider the following grammar:

$$\begin{aligned} S &::= X \mid Y \mid Z \\ X &::= a \mid aX \mid aaXb \\ Y &::= aaabb \mid aYb \mid aaYb \\ Z &::= b \mid aZb \mid Zb \end{aligned}$$

b $(a + b)^* = \{(a^n b^n)^n \mid n \geq 1\}$

This is not a CFL. To see this, given $k \in \mathbb{N}$, try pumping $(a^k b^k)^k$.

c $\{ww^R w \mid w \in (a + b)^*\}$

This is not a CFL. To see this, given $k \in \mathbb{N}$, try pumping $a^n b^n b^n a^n a^n b^n$.

d $\{b_i \# b_{i+1} \mid b_i \text{ represents } i \text{ in binary}\}$

This is not a CFL. To see this, given $k \in \mathbb{N}$, try pumping $1^n 0 1^n \# 1^n 10^n$.

Exercise 3. Pushdown automata

The deterministic PDA (DPDA) is not equivalent to the nondeterministic PDA (NPDA). Consider the language

$$L = \{0^n 1^n \mid n \geq 1\} \cup \{0^n 1^{2n} \mid n \geq 1\}$$

a Show that L is a CFL.

Consider the following grammar:

$$\begin{aligned} S &::= A \mid B \\ A &::= 0A1 \mid 01 \\ B &::= 0B11 \mid 011 \end{aligned}$$

b Prove that L is not accepted by a DPDA.

Suppose that L is accepted by an empty-stack DPDA. Then the stack must be empty both after reading $0^n 1^n$ and after reading $0^n 1^{2n}$. Now, given n , let q_n be the state of the machine after reading $0^n 1^n$. As a DPDA can only have a finite number of states, $\exists n, m \in \mathbb{N}$ s.t. $q_n = q_m$. But this implies that $0^n 1^{n+m}$ and $0^m 1^{n+m}$ must also leave the stack empty. It follows that these two strings are accepted by our DPDA despite the fact that they are not accepted by L . We can thus conclude that there doesn't exist a DPDA that accepts L .

Exercise 4. Regular languages

Show that if L is a CFL over a one-symbol alphabet, then L is regular.

Easy way: Use Parikh's Theorem (from the book).

Harder way: Let L be such a CFL. Suppose we are given $k \in \mathbb{N}$. Then consider $z \in L$ s.t. $|z| \geq k$. Let $z = uvwxy$. Letting $q = |vx| > 0$ and $p = |z| - q$, we have $|uv^iwx^iy| = p + iq$. Thus, $\forall z \in L$ long enough, $0^{p+iq} \in L \forall i$.

As order does not matter in strings over a one-symbol alphabet, there are at most n words in L that have length $< n$. For each of these strings let $s = p \bmod q$. Then there are, at most, n^2 possible combinations of s and q .

As $\forall m > ns.t. 0^m \in L, \exists p, q$ s.t. $0^m \in \{0^{p+iq} \mid i \geq 0\}$, it follows that L can be generated from a finite number of its strings and thus L is regular.