CS 286a | **HW 5: End Game** | **Guru: Adam Wierman**

Assigned: 11/21/08 | Due: 12/1/08, my office, 5pm

I encourage you to discuss these problems with others, but you need to write up the actual solutions alone. At the top of your homework sheet, please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you. Also, please limit your use of the web as it is difficult to develop good problems, and so enough search will probably lead you to find the solutions online.

1 Busy periods [15 points]

Ideally you should solve this problem by Monday Nov 24, since we will use busy periods fundamentally in the upcoming lectures.

A busy period starts when the server becomes busy and ends when it goes idle. It turns out that understanding the distribution of the length of a busy period is fundamental to the study of many scheduling policies, as we will see in the next few lectures.

Consider an M/GI/1 queue and let B denote the length of a busy period and let $B(x)$ denote the length of a busy period conditional on the first job in the busy period having size x.

(a) Use renewal-reward arguments to derive the mean length of a busy period, $E[B]$.

(b) Derive the Laplace transform of $B(x)$ in terms of the Laplace transform of B.

(c) Use the Laplace transform of $B(x)$ to derive the following recursive formula for the Laplace transform of B:

$$\widetilde{B}(s) = \widetilde{X}(s + \lambda - \lambda\widetilde{B}(s)).$$

(d) Take derivatives of the Laplace transform in order to calculate $E[B]$ and $E[B^2]$.

(e) Contrast $E[B]$ with $E[T_Q^{FCFS}]$ when $E[X^2]$ is large/small. (Use the phrase "inspection paradox.")

2 The distribution of Excess [15 points]

In class we used renewal-reward theory to derive the mean excess. It turns out we can use the same ideas to derive the whole distribution of excess.

(a) Use renewal-reward to derive $Pr(\mathcal{E}(X) > t)$.

(b) Use the above to derive the p.d.f. of excess.

(c) Now calculate the Laplace transform of excess (which will be in terms $\widetilde{X}(s)$).

(d) Derive a formula for the ith moment of excess.

3 Learning from the transform [25 points]

In lecture we derived the Laplace transform for T^{FCFS} and T_Q^{FCFS}, but we didn't "learn" much from them yet. In this problem, you'll see how powerful transforms can be.

(a) *Deriving moments:* The easiest thing to do with a transform is always to derive the moments.

Use the transform to derive the first and second moment of T_Q.

(b) *Obtaining the distribution:* In some special cases it is possible to determine the c.d.f. from the Laplace transform. This is referred to as "inverting the transform" even though this is not the same as the traditional meaning of invert.

Consider $\lambda = 1$ and job sizes that are a 2-phase Erlang with $E[X] = 1/3$. So

$$\widetilde{X}(s) = \left(\frac{6}{6+s}\right)^2.$$

Prove that $F_T(t) = \frac{8}{5}(1 - e^{-3t}) - \frac{3}{5}(1 - e^{-8t})$. (Note that this is for T, not T_Q.)

(c) *Asymptotic analysis:* In the general case, when the Laplace transform can't be used to calculate the distribution exactly, it is often possible to determine the distribution in some asymptotic regime. In this case, we'll consider the heavy-traffic regime, i.e., $\rho \to 1$.

Prove that as $\rho \to 1$,

$$(1-\rho)T_Q \sim Exponential(1/E[\mathcal{E}(X)])$$

Hint: You want to study $\widetilde{T}((1-\rho)s)$ as $\rho \to 1$.

(d) **[Extra Credit]** *Numerical inversion:* If you really want to attain the c.d.f. from the Laplace transform and you can't do it by hand, you can still do it numerically.

Use the EULER method of transform inversion to numerically calculate the c.d.f. for the situation from part (b). A description of the EULER method can be found on Ward Whitt's site: `http://www.columbia.edu/~ww2040/6711F08/LaplaceInversionJoC95.pdf` Ward Whitt is one of the world experts of transform inversion.

4 Servers that sleep [20 points]

Let's return to the example of setup costs that we discussed in lecture, but now we will consider an M/GI/1 queue with setup costs. In particular, consider the setting where once a server becomes idle it goes into power-saving mode and takes I time to "wake-up" when a new arrival occurs.

(a) Use the same technique as we used to derive the transform of the standard M/GI/1 model in order to derive

$$\widehat{N}(z) = \frac{\pi_0(z\widehat{A}_{X+I}(z) - \widehat{A_X}(z))}{z - \widehat{A_X}(z)}$$

(b) Derive π_0.

(c) Derive $\widetilde{T}(s)$ using the above.

(d) **(Extra Credit)** Derive $E[T]$ and contrast it with $E[T]^{M/GI/1}$.

5 One fast server versus two slow servers [15 points]

We've returned a few times to the question of whether it is better to have one fast server of two slower servers of with half the speed. Now, we can study this question in the case of non-exponential job sizes for the first time.

Throughout assume that we have Poisson arrivals.

(a) Recall from previously in the class – Is one fast server or two slow servers better when job sizes are exponentially distributed?

(b) Now, suppose we have the following, high variability, distribution for job sizes.

$$X = \begin{cases} 0.01, & \text{with probability } 100/101 \\ 1, & \text{with probability } 1/101 \end{cases}$$

The two systems we want to compare are:

(i) A single "fast" M/GI/1/FCFS.

(ii) Two "slow" M/GI/1/FCFS queues where arrivals of size 0.01 are sent to server 1 and arrivals of size 1 are sent to server 2.

A job which requires s seconds on the fast machine takes $2s$ seconds on a slow machine.

Compute $E[T_Q]$ in both cases and compare the result. Would the comparison change if X was less variable? If so, give an example. Would the comparison change if we considered $E[T]$?

6 Extending Little's Law [10 points]

In class we argued that $N = A_T$ in distribution. This is because the number left behind when a job departs is exactly the same as the number that arrived during the time the job was in the system. This relationship is enough to prove a specialization of Little's Law for higher moments, which we discussed in class earlier.

Prove that $E[N(N-1)\cdots(N-k+1)] = \lambda^k E[T^k]$ under any policy for which $N = A_T$ in distribution. What assumptions are made about the underlying system by this argument?