CS 286a — **HW 2: Practice with DTMCs** — **Guru: Adam Wierman**

Assigned: 10/6/08 — Due: 10/17/08 10:30am

I encourage you to discuss these problems with others, but you need to write up the actual solutions alone. At the top of your homework sheet, please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you. Also, please limit your use of the web as it is difficult to develop good problems, and so enough search will probably lead you to find the solutions online.

Note before you begin

This is a long problem set. Start early – do not put it off until the last minute!

1 Playing in "Transform world" [Optional]

In this class we will often be using the Laplace transform. This problem should introduce you to the basic properties you'll need. If this is hard for you, please come see me and I'll give you some background reading.

The Laplace transform of a random variable X is defined as

$$L_X(s) = E[e^{-sX}].$$

In this problem, you'll refresh your memory of some of the basic properties of Laplace transforms.

(a) Let $X \sim Exponential(\lambda)$. Show that $L_X(s) = \frac{\lambda}{\lambda+s}$.

(b) Let X be random variable with continuous p.d.f. $f(x)$. Show that $E[X^n] = (-1)^n \left.\frac{d^n L_X(s)}{ds^n}\right|_{s=0}$.

(c) Let X and Y be independent random variables. Show that $L_{X+Y}(s) = L_X(s)L_Y(s)$.

2 Hazardous material [10 points]

Define the hazard rate (failure rate) of a continuous random variable X having p.d.f. $f(x)$ and c.d.f. $F(x)$ as

$$h(x) = \frac{f(x)}{\bar{F}(x)},$$

where $\bar{F}(x) = 1 - F(x)$.

The hazard rate is an important quantity that will comes up over and over again. To interpret $h(x)$ let us think of X as the lifetime of some item, and suppose that the item has survived for t time and we would like to understand the likelihood that it will not survive another dt time (i.e. the rate at which it will fail right now). Then we have

$$Pr(X \in (t, t+dt) | X > t) = \frac{Pr(X \in (t, t+dt), X > t)}{Pr(X > t)} = \frac{Pr(X \in (t, t+dt))}{Pr(X > t)} = \frac{f(t)}{\bar{F}(t)}$$

Now, let's prove a few important properties of $h(t)$.

(a) Show that the exponential distribution has a constant hazard rate.

(b) Prove that the hazard rate uniquely determines the distribution. Specifically, prove that

$$\bar{F}(x) = e^{-\int_0^x h(t)dt}.$$

Use this result to argue that the exponential distribution is the *unique* continuous distribution with a constant hazard rate.

(c) If X_1 and X_2 are independent, non-negative, continuous random variables, show that

$$Pr(X_1 < X_2 | \min(X_1, X_2) = t) = \frac{h_1(t)}{h_1(t) + h_2(t)}.$$

3 DTMC Warmup [20 points]

(a) For each of the following state whether the chain is irreducible, aperiodic, and/or positive recurrent.

(i) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Consider the following probability transition matrices

$$\mathbf{P}^{(1)} = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 2/3 & 0 & 1/3 & 0 \end{pmatrix}$$

$$\mathbf{P}^{(2)} = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix}$$

For each of these,

(i) Draw the corresponding Markov chains.

(ii) Solve for the stationary probabilities. First, try to use the local balance equations, and if they don't work use the balance equations.

(iii) For those chain(s) there the local balance equations worked, explain why it makes sense that for all states i, j in the chain, the rate of transitions from i to j should equal the rate of transitions from j to i.

4 Walking around a graph [25 points]

Let $G = (V, E)$ be a finite, undirected graph where V is the vertex set and E is the edge set.

(a) Consider particle taking a random walk over a connected graph where at each vertex the particle is equally likely to follow each of the edges.

   (i) When is the corresponding Markov chain aperiodic?

   (ii) What is the stationary distribution for the position of the particle?

(b) We will use the above analysis to give a simple algorithm for determining if two vertices s and t are connected in G. (G is no longer assumed to be connected.)

   Note that we could of course determine if s and t are connected using a breadth/depth first search, but such an algorithm would require $\Omega(|V|)$ space. We'll do much better than this.

   Here is the algorithm: Start a random walk from s. If the walk reaches t within $4|V|^3$ steps, return that there is a path. If not, return that there is no path.

   Here is an outline of the analysis of the algorithm for you to fill in the details of:

   (i) Let $h_{u,v}$ denote the expected number of steps to go from u, v, i.e., the "hitting time."

   (ii) Prove that $h_{u,u} = 2|E|/d(u)$, where $d(u)$ is the degree of u.

   (iii) Prove that if $(u, v) \in E$, then $h_{u,v} < 2|E|$.

   (iv) Let $C(G)$ denote the cover time of a graph. That is, let $C(G)$ be the maximum over all v of the expected time to visit all nodes in G via a random walk starting at v. Prove that $C(G) \leq 4|V| \cdot |E|$.

   (v) Show that the algorithm will return the correct answer with probability 1/2. Since the algorithm can only err by saying the graph is connected when it is not, we can then repeat the algorithm n times to attain the correct answer with probability $1 - 1/2^{|V|}$.

   (vi) Argue that the space used is $O(\log(|V|)$. Interestingly, this is the minimal space requirement possible.

5 Randomized chess [15 points]

Disclaimer: The last three parts of this problem depend on the result from Problem 4(a).

In this problem we will study the behavior of various chess pieces as they move randomly around the board. In case you are not familiar with chess, all you need to know for this problem is the following. The game is played on an board divided into 64 squares (8x8) that alternate from white to black. There are many different types of pieces that each move in a different way. The three we talk about are the king, bishop, and knight. The king can move one square in any direction (including the diagonal). The bishop can move any number of squares, but only in the diagonal directions. Finally, the knight moves in an L-shape. That is, the knight can move 2 squares to either side and one square up or down. Or, the knight can move two squares up or down and 1 square left or right.

(a) You are given an empty 8x8 chessboard with a lone king placed in one corner. At each time step, the king will make a uniformly random legal move. Is the corresponding Markov chain for this process (i) irreducible? (ii) aperiodic?

(b) What if a bishop is used instead?

(c) What if a knight is used instead?

(d) Now, take advantage of Problem 4(a) to calculate the expected time for the king to return to the corner.

(e) Do the same for the bishop.

(f) Do the same for the knight.

6 Finite state DTMCs [15 points]

(a) Prove the following two class property theorems.

   (i) Null-recurrence is a class property, i.e., if i is null-recurrent and i communicates with j then j is null-recurrent.

   (ii) Positive-recurrence is a class property, i.e., if i is positive-recurrent and i communicates with j then j is positive-recurrent.

(b) Prove that in a finite state, aperiodic, irreducible DTMC, all states are positive recurrent; and thus that the limiting distribution exists and has $\pi_i > 0$ for all i.

7 A drunkard's walk [15 points]

In this problem we will return to our drunken man making a random walk.

(a) Let us now consider a random walk in a two dimensional grid, i.e., the infinite plane. The drunkard is equally likely to walk one unit up, down, left, or right at each point. If he starts at the origin, will he always return there? or is there some probability he will wander off without ever returning to the origin?

   A possible approach is:

   (i) Begin by arguing that $P_{00}^{2n} = \left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^2$.

   (ii) Apply Stirling's Formula conclude whether or not $\sum_{n=0}^{\infty} P_{00}^{n}$ converges.

(b) Now, consider a drunkard making a random walk in a three dimensional space. (Maybe a drunk bird?) If he starts at the origin, will he always return there? or is there some probability he will wander off without ever returning to the origin?

   The same proof approach as above will work, though it is more complicated.

   (i) **Verifying this part is extra credit.** Begin by arguing that

$$P_{00}^{2n} \leq \frac{n!}{((n/3)!)^3 2^{2n} 3^n} \binom{2n}{n}.$$

   (ii) Apply Stirling's Formula to conclude whether or not $\sum_{n=0}^{\infty} P_{00}^{n}$ converges.