Efficiency and Revenue in Certain Nash Equilibria of Keyword Auctions

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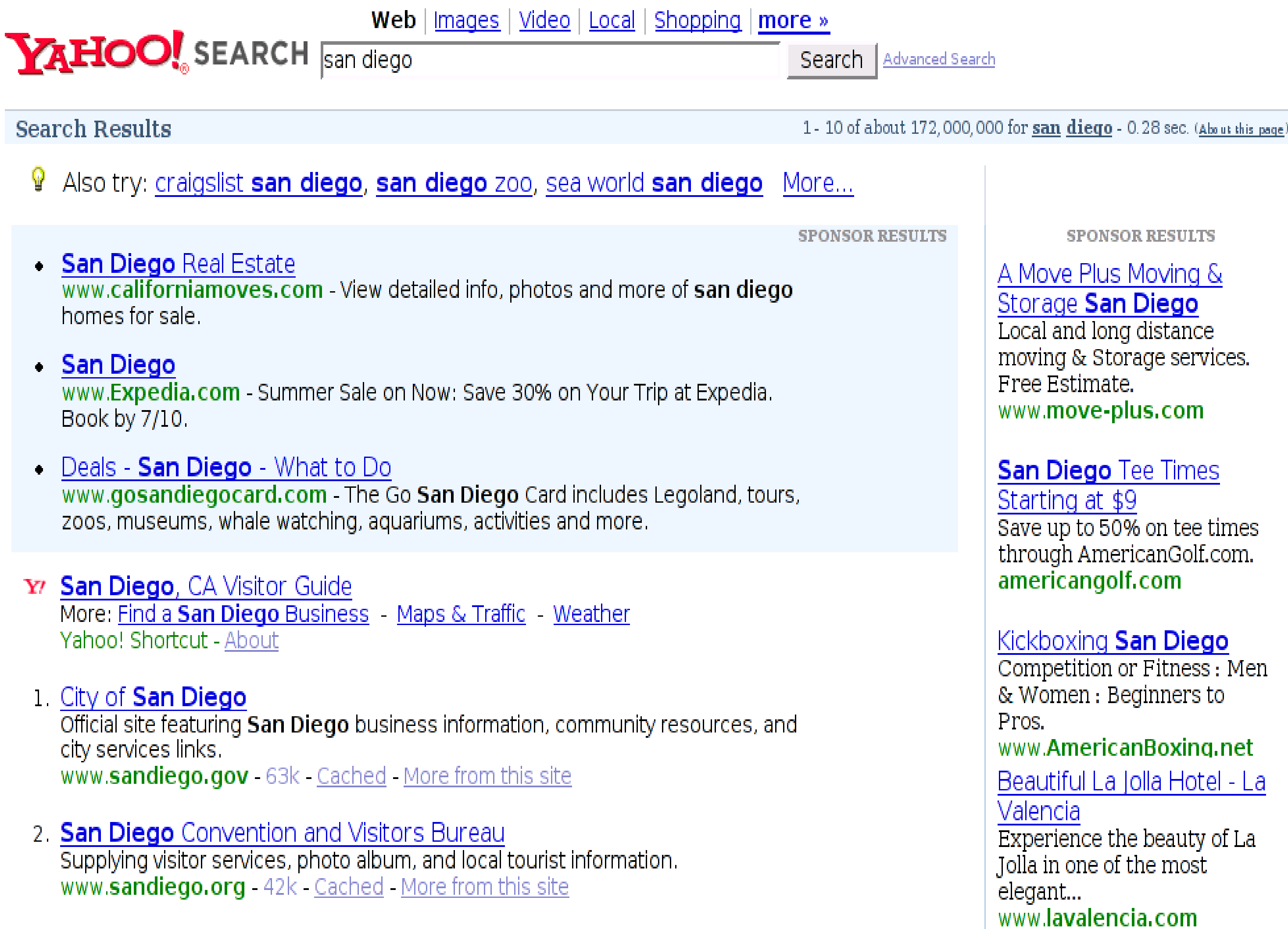
Sponsored Search



Outline

- Model for keyword auctions.
- Efficiency in pure-strategy Nash equilibrium.
  - Necessary conditions for equilibrium.
  - Worst-case bound on efficiency.
- Revenue in symmetric equilibrium.
  - General case.
  - Restricted family of weights.
  - Efficiency and relevance considerations.

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Model

- K positions, N bidders.
- The click-through rate of bidder s in positions t is $e_s x_t$, i.e. *separable* into
  1. advertiser effect (or *relevance*) e_s
  2. position effect x_t $(x_1 > x_2 > \ldots > x_K)$.
- Bidder s has per-click value of v_s.
- If bidder s obtains slot t at price of p per click, utility is

$$e_s x_t (v_s - p),$$

  i.e. quasi-linear.

Auction Rules

- The auctioneer assigns a weight w_s to each bidder s [Aggarwal et al. '06].
- Bidders submit bids (reported values) b_s.
- Bidders are ranked in order of decreasing *score* $w_s b_s$.
- Bidder s pays per click the lowest bid necessary to maintain its position:

$$w_s b_s \geq w_{s+1} b_{s+1} \quad \Rightarrow \quad b_s \geq \frac{w_{s+1}}{w_s} b_{s+1}$$

- "Yahoo model": $w_s = 1$.
- "Google model": $w_s = e_s$.

Efficient Ranking

- A bidder's *true score* is $r_s = w_s v_s$.
- An allocation of slots to bidders $\sigma : K \to N$ maximizes the objective

$$\sum_t x_t w_{\sigma(t)} v_{\sigma(t)}$$

  if and only if bidders are ranked in decreasing order of true score.
- Follows easily from the fact that $x_1 > x_2 \ldots > x_K$.
- If $w_s = e_s$, the objective is the *total value*.

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Pure-Strategy Nash Equilibrium

- Ad-hoc justification:

  bidders constantly update their bids until they find their current position is preferred to any other, given others' bids.

- An allocation and vector of bids constitute a pure-strategy Nash equilibrium if, for each slot s,

$$
\begin{aligned}
& e_s x_s\left(v_s - \frac{w_{s+1}}{w_s} b_{s+1}\right) \geq e_s x_t\left(v_s - \frac{w_{t+1}}{w_s} b_{t+1}\right) \quad (t > s) \\
\Leftrightarrow \quad & x_s(w_s v_s - w_{s+1} b_{s+1}) \geq x_t(w_s v_s - w_{t+1} b_{t+1}) \quad (t > s)
\end{aligned}
$$

  and

$$
x_s(w_s v_s - w_{s+1} b_{s+1}) \geq x_t(w_s v_s - w_t b_t) \quad (t < s)
$$

Partial Characterization

- Can have a multiplicity of Nash equilibria, in terms of both bid vectors and allocations.
- What allocations can arise in Nash equilibrium?

Lemma 1 *An allocation can arise in pure-strategy Nash equilibrium only if*

$$r_i \geq \frac{x_i}{x_{i+1}} \frac{x_{i+1} - x_j}{x_i - x_j} r_j$$

for $1 \leq i \leq K - 2$ *and* $j \geq i + 2$*.*

- Proof sketch: Farkas lemma.
- Complete characterization known for the case where $N = 3$ [Börgers et al., '07].

Bound on Efficiency

Example: 5 bidders, number them such that $r_1 \geq r_2 \geq \ldots \geq r_5$.

$$
\begin{aligned}
& \frac{x_1 r_3 + x_2 r_2 + x_3 r_1 + x_4 r_5 + x_5 r_4}{x_1 r_1 + x_2 r_2 + x_3 r_3 + x_4 r_4 + x_5 r_5} \\
\geq \quad & \min \left\{ \frac{x_1 r_3}{x_1 r_1}, \frac{x_2 r_2}{x_2 r_2}, \frac{x_3 r_1}{x_3 r_3}, \frac{x_4 r_5}{x_5 r_5}, \frac{x_5 r_4}{x_4 r_4} \right\} \\
= \quad & \min \left\{ \frac{x_1 r_3}{x_1 r_1}, \frac{x_5 r_4}{x_4 r_4} \right\} \\
\geq \quad & \min \left\{ \frac{x_1}{x_2} \frac{x_2 - x_3}{x_1 - x_3}, \frac{x_5}{x_4} \right\}
\end{aligned}
$$

In general, we can give a lower bound of

$$
L = \min_{i=1,\ldots,N-1} \min \left\{ \frac{x_i}{x_{i+1}} \frac{x_{i+1} - x_{i+2}}{x_i - x_{i+2}}, \frac{x_{i+1}}{x_i} \right\}
$$

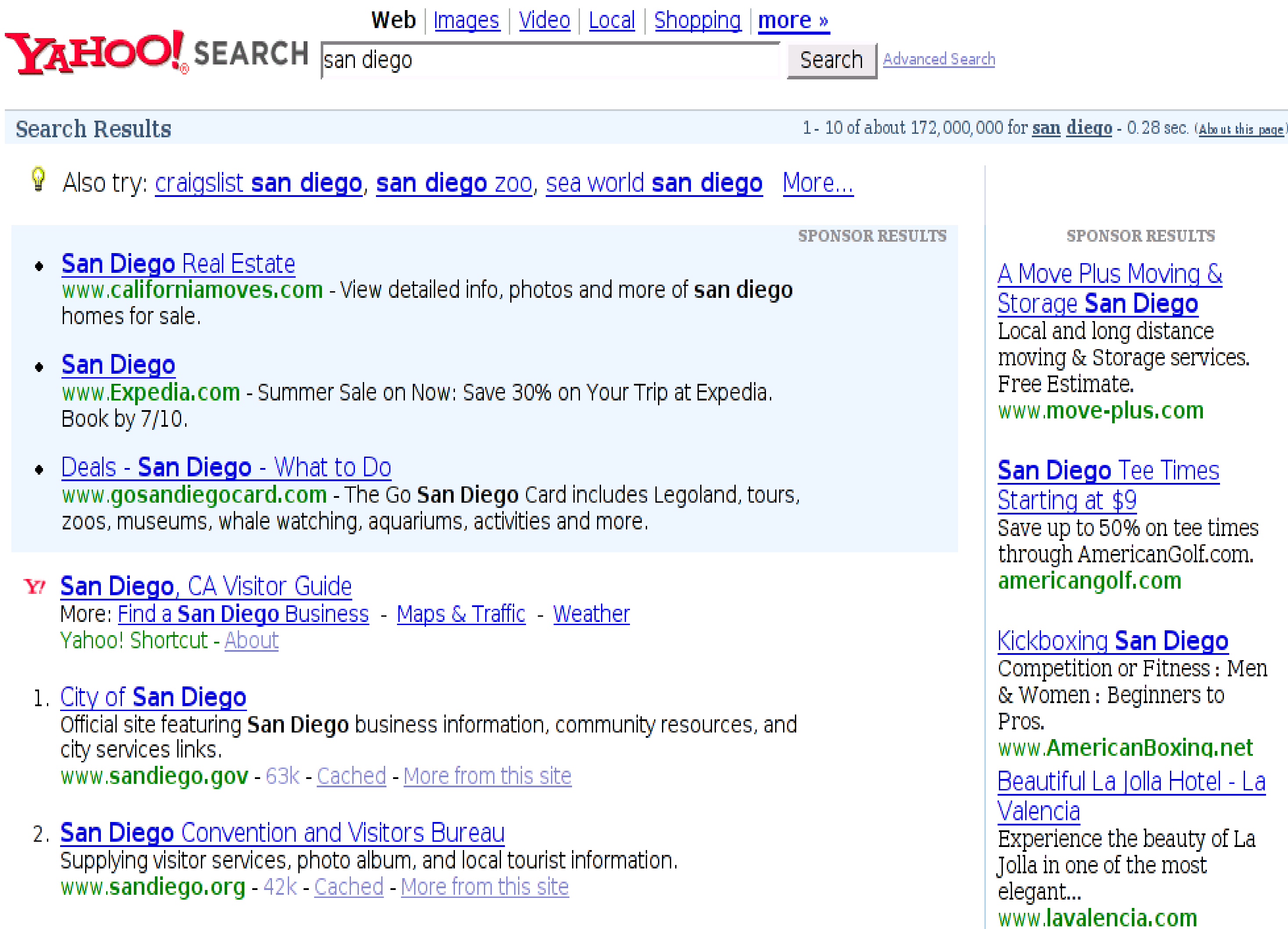
Exponential Decay

- For the exponential decay model, $x_t \sim 1/\delta^t$ for $\delta > 1$, we have

$$L = \min\left\{\frac{\delta}{1+\delta}, \frac{1}{\delta}\right\}$$

  - $\delta = 1.428$ [Feng et al., '06], $L = 0.6$.
  - $\delta = 1.5$ [Börgers et al., '07], $L = 0.59$.
- My own estimates on a keyword, without exponential decay, $L = 0.07$...
- Need to pay particular attention to ordering at the top, and at breaks in the ad display.

Slot Layout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Symmetric Equilibrium

- To analyze revenue, assume bidders are playing a
  "locally envy-free equilibrium" [Edelman et al. '06]
  "symmetric equilibrium" [Varian '06].
- An allocation and vector of bids constitute a symmetric equilibrium if, for slots s, t,

$$e_s x_s (v_s - \frac{w_{s+1}}{w_s} b_{s+1}) \geq e_s x_t (v_s - \frac{w_{t+1}}{w_s} b_{t+1})$$

$$\Leftrightarrow \quad x_s (w_s v_s - w_{s+1} b_{s+1}) \geq x_t (w_s v_s - w_{t+1} b_{t+1})$$

Properties

- Implies that complementary slackness conditions for the assignment problem hold.
- In symmetric equilibrium, bids are such that bidders are ranked in order of decreasing true score.
  - $w_s v_s \geq w_t v_t \Leftrightarrow w_s b_s \geq w_t b_t$.
  - Maximizes objective $\sum_t x_t w_{\sigma(t)} v_{\sigma(t)}$.
  - If $w_s = e_s$, symmetric equilibrium is efficient.
- Set of symmetric equilibrium bids forms a lattice [Shapley and Shubik, '72].
  - There exist minimum and maximum bid vectors.
  - Select the minimum element, to optimize a lower bound on revenue.
  - When $w_s = e_s$, minimum element gives Vickrey payments [Leonard, '83].

Payments in Symmetric Equilibrium

Let

$$y_{st}(e,v) = \begin{cases} 1 & \text{if bidder } s \text{ gets slot } t \\ 0 & \text{otherwise} \end{cases}$$

The total payment of bidder s in minimum symmetric equilibrium is

$$\begin{aligned} & \sum_{t=s}^{K} \frac{w_{t+1}}{w_s} e_s (x_t - x_{t+1}) v_{t+1} \\ = \quad & e_s \sum_{t=1}^{n} x_t \left[v_s y_{st}(e,v) - \int_0^{v_s} y_{st}(e,\tau,v_{-s})\, d\tau \right]. \end{aligned}$$

Very similar to [Myerson '81]'s analysis of the single-item case.

General Problem Formulation

$$\begin{aligned} \max_{w} \quad & \int_{[0,1]^N} \int_{[0,\infty]^N} \sum_{s=1}^{N} \sum_{t=1}^{K} x_t e_s \psi_s(e_s, v_s) y_{st}(e,v) f(e,v)\, dv\, de \\ \text{s.t.} \quad & \sum_t y_{st}(e,v) \le 1 \qquad \forall s, e, v \\ & \sum_s y_{st}(e,v) \le 1 \qquad \forall t, e, v \end{aligned}$$

Would like to rank bidders according to their "virtual scores" $e_s \psi(e_s, v_s)$.

$$\psi(e_s, v_s) = v_s - \frac{1 - F(v_s|e_s)}{f(v_s|e_s)}$$

But we are restricted to ranking according to scores of the form

$$g(e_s)v_s + h(e_s).$$

Restricted Family of Weights

We restrict our attention to weights

$$w_s = e_s^q$$

where $q \in (-\infty, +\infty)$.

- Yahoo model: $q = 0$.
- Google model: $q = 1$.

Recall the equilibrium payment:

$$\sum_{t=s}^{K} \left(\frac{e_{t+1}}{e_s} \right)^q e_s (x_t - x_{t+1}) v_{t+1}$$

Efficiency and Relevance

Total relevance is $\sum_{t=1}^{K} e_t x_t$.

Proposition 1 *Total relevance is non-decreasing in q.*

Intuition: the impact of relevance on the score increases relative to the bid as q increases.

Total value (efficiency) is $\sum_{t=1}^{K} e_t x_t v_t$.

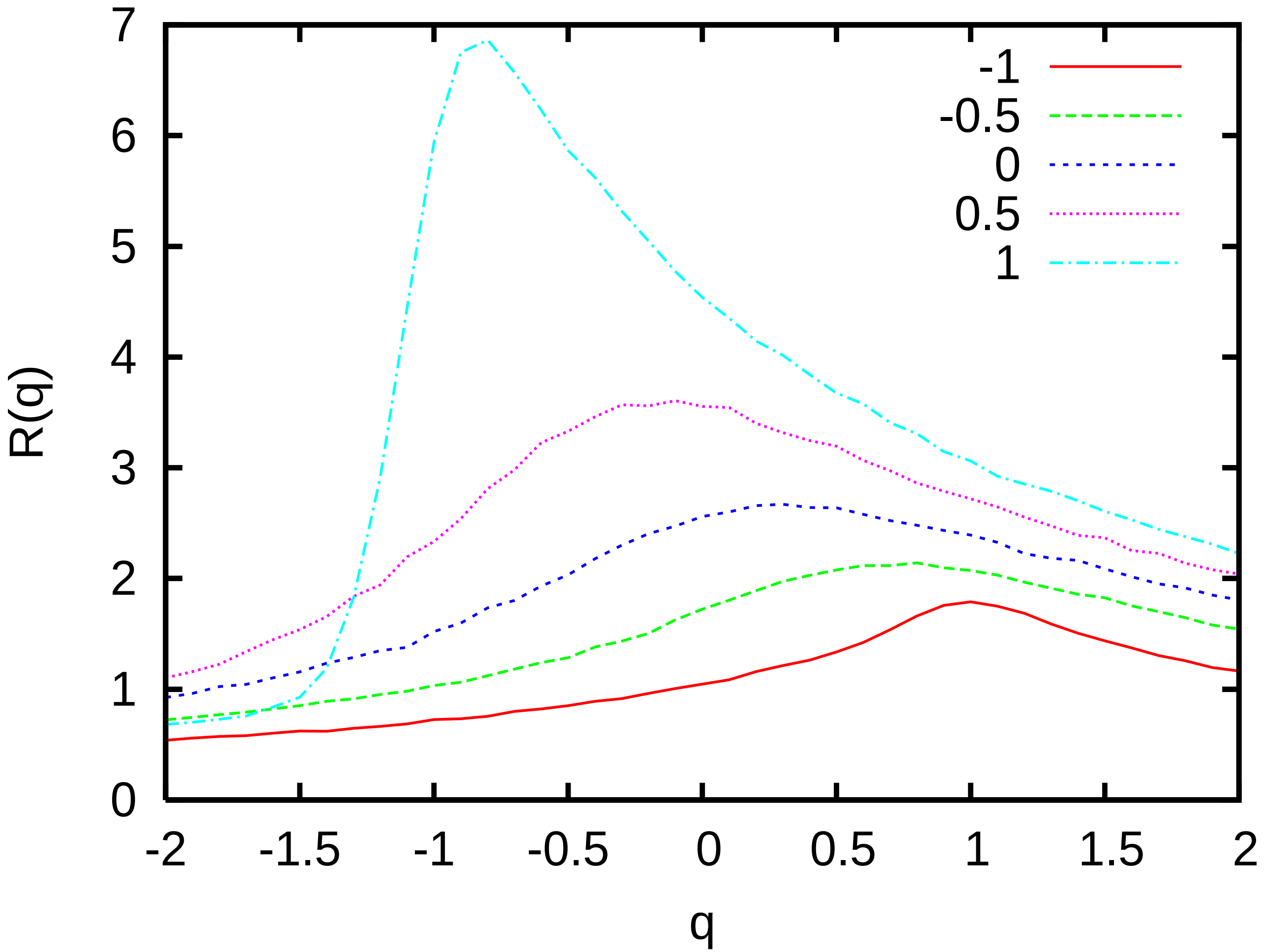
Proposition 2 *Total value is non-decreasing in q for $q \leq 1$ and non-increasing in q for $q \geq 1$.*

Intuition: at $q = 1$, efficiency is maximized in equilibrium.

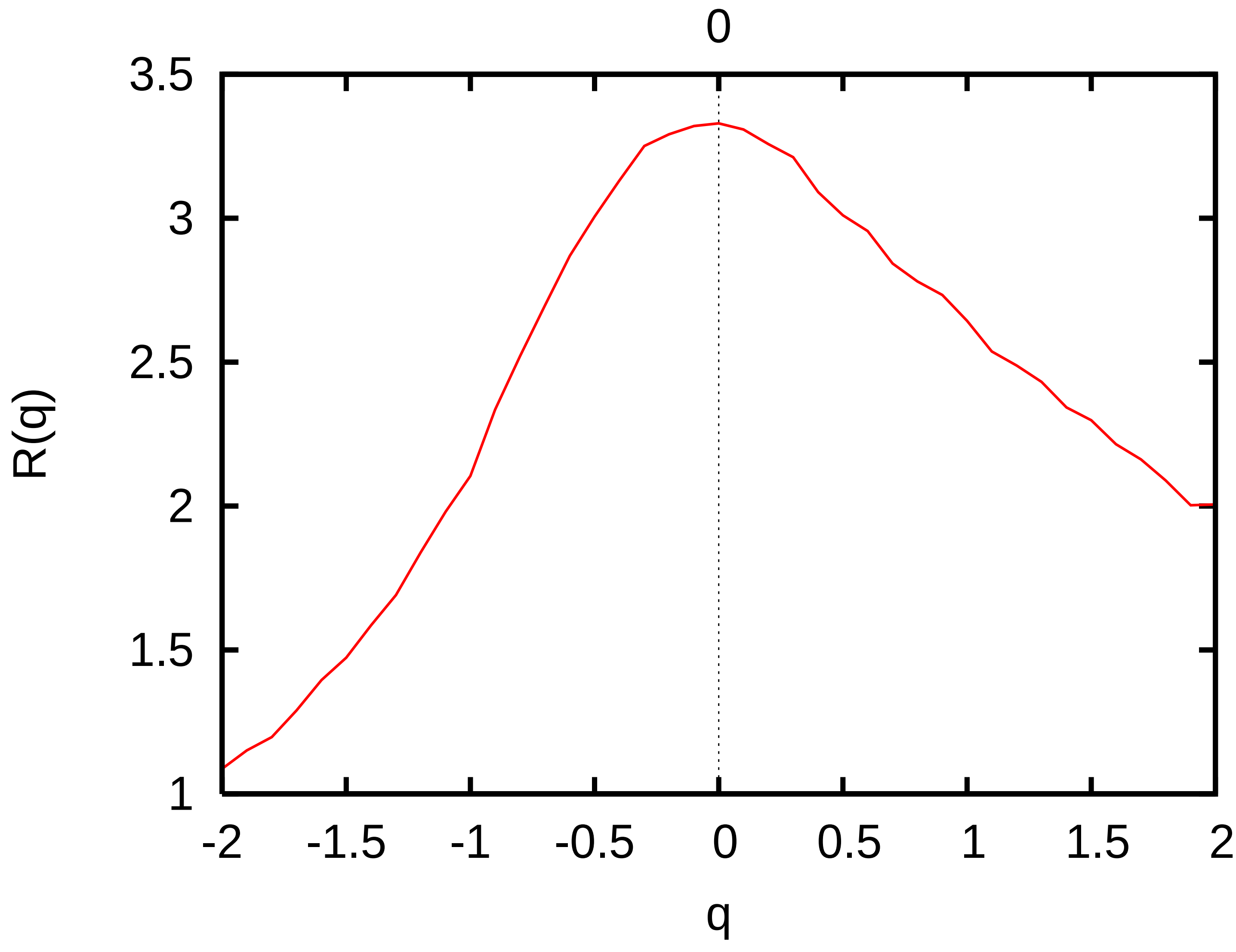
Simulations

- Obtained bid and click-through rate data for advertisers bidding on a high-volume keyword in summer of 2006.
- Bidder values estimated by deriving bounds according to symmetric equilibrium [Varian '06].
- Marginal distributions:
  - Value: Lognormal, $\mu = 0.35$ and $\sigma = 0.71$.
  - Relevance: Beta, $a = 2.71$ and $b = 25.43$.
- Spearman correlation between value and relevance was in $[0.36, 0.55]$ over a month.
- Modeled dependence between value and relevance with a Gaussian copula.

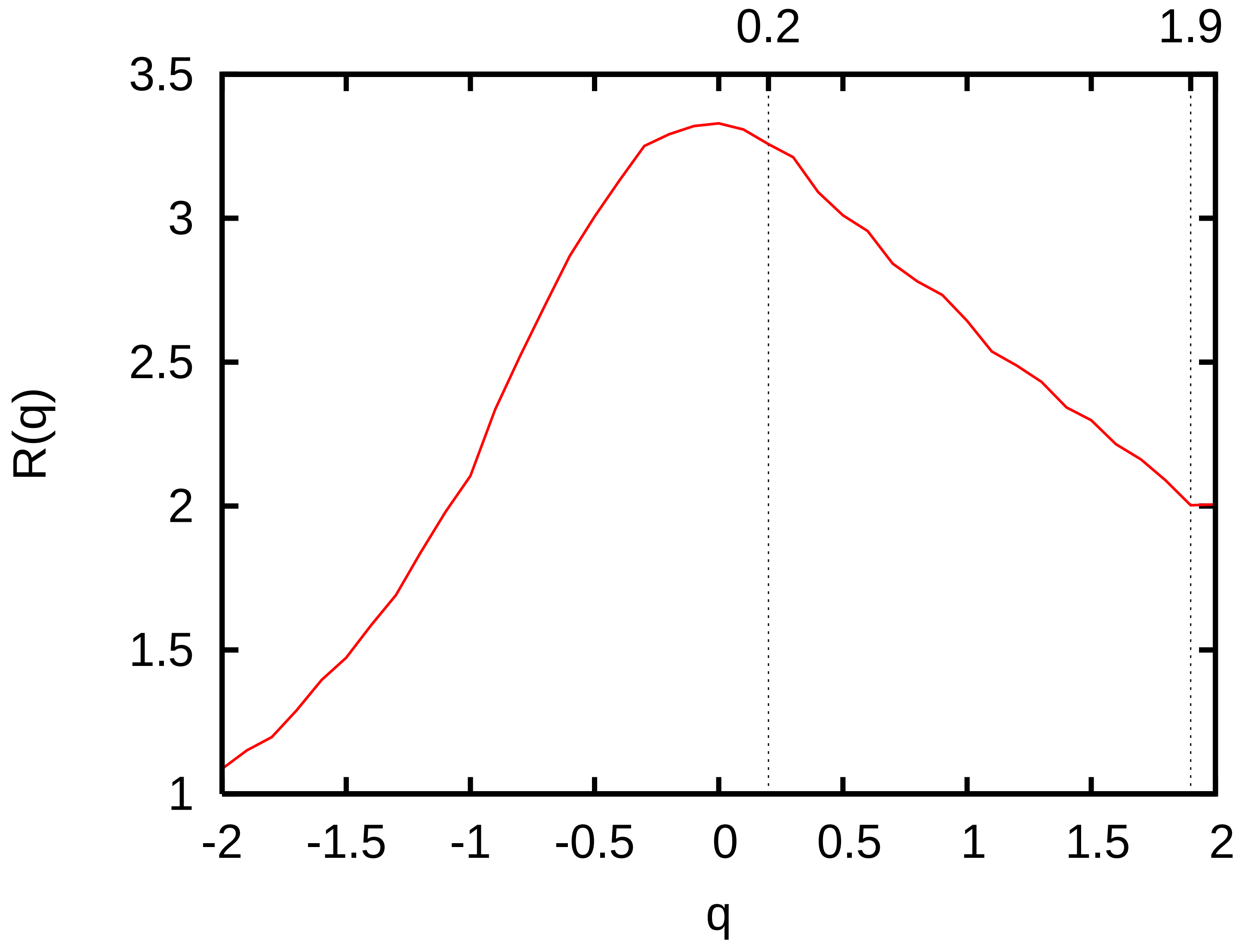
Revenue Effect of Correlation



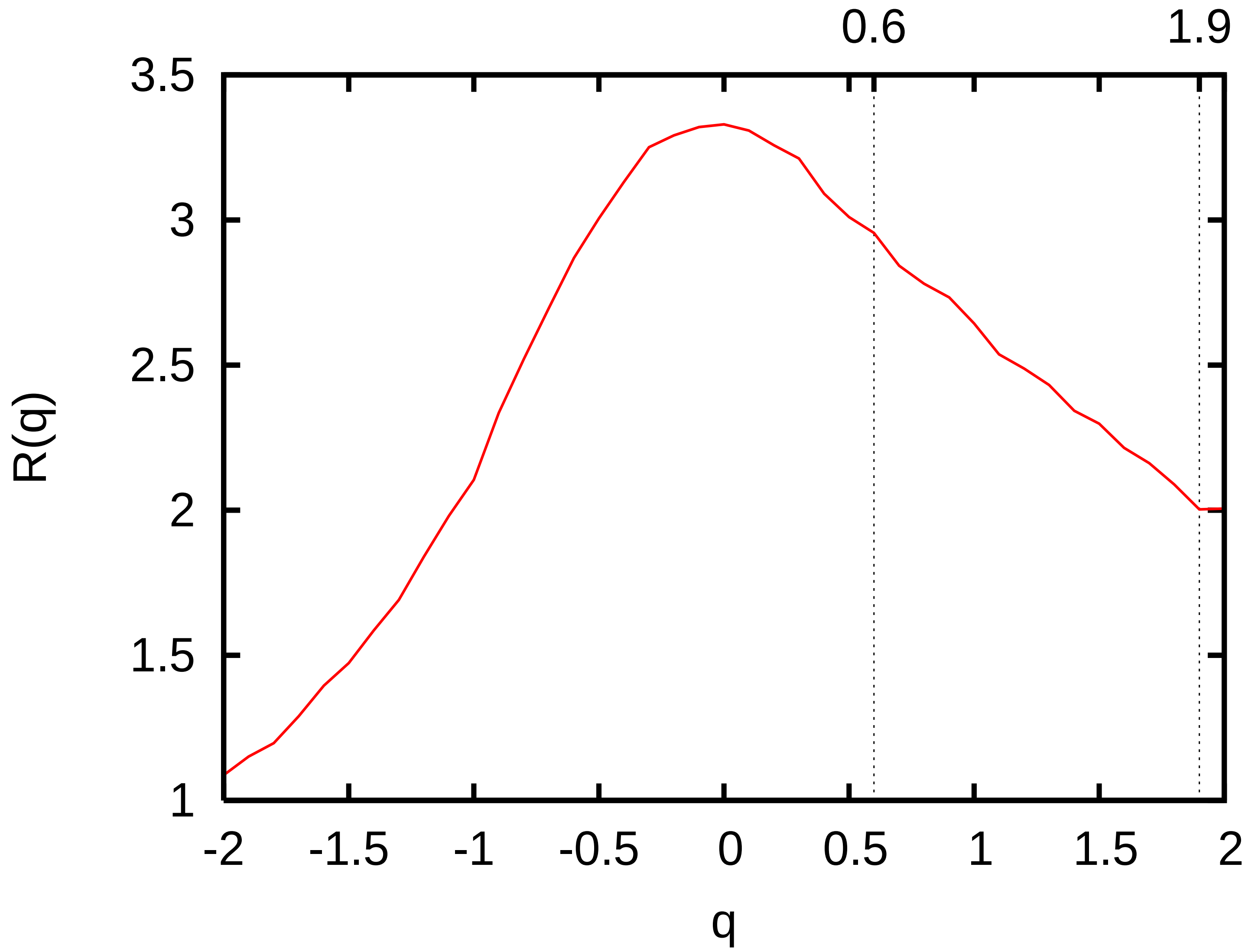
Correlation of 0.4



Efficiency Bounds



Relevance Lower Bound



Bidding Credits

Suppose bidder s only pays a fraction $c_s \in [0,1]$ of the price he faces. The symmetric equilibrium constraints are

$$e_s x_s(v_s - c_s \frac{w_{s+1}}{w_s} b_{s+1}) \geq e_s x_t(v_s - c_s \frac{w_{t+1}}{w_s} b_{t+1})$$

$$\Leftrightarrow \quad x_s(\frac{w_s}{c_s} v_s - w_{s+1} b_{s+1}) \geq x_t(\frac{w_s}{c_s} v_s - w_{t+1} b_{t+1})$$

Letting $w'_s = \frac{w_s}{c_s}$ and $b'_s = c_s b_s$, we get

$$x_s(w'_s v_s - w'_{s+1} b'_{s+1}) \geq x_t(w'_s v_s - w'_{t+1} b'_{t+1})$$

Back to the original symmetric equilibrium inequalities!

- Revenue will be the same as if we had used weights w'.
- To go from Google-model revenue to Yahoo-model revenue, set credits to $c_s = e_s$.

Summary

- With correlation of 0.4, $q = 0$ is optimal, yielding 25% more revenue than $q = 1$.
- Imposing a bound of 5% loss in efficiency and relevance from baseline of $q = 1$, $q = 0.6$ is optimal with 11% improvement in revenue.
- Optimal reserve score is 0.2—it gives 8% increase in revenue, but 13% efficiency loss and 26% relevance loss.
- In theory, same effect could be achieved with bidding credits.

Conclusions

- Pure-strategy Nash equilibria of keyword auctions should be quite efficient.
- Open question as to whether “swaps” are a problem.

- Optimal keyword auction design problem can be formulated as a mathematical program.
- Open question as to how to solve the program for general weighting schemes.
- Changing exponent q on advertiser effect can improve revenue, depending on correlation between value and relevance.
- Simulations suggest this approach can be more effective than using a reserve score.