The Price of Anarchy Revisited

- Objective function: Average latency
- For two-link, two node network with arbitrary latency functions, the price of anarchy, ρ, was unbounded
- For linear latency functions, $\rho \leq 4/3$ independent of the network topology
- The Braess ratio β is also bounded by $4/3$ for linear latency functions and is bounded by $n/2$ for arbitrary latencies
- Other possible objectives?

The Price of Anarchy Revisited

- Fairness: Objective is maximum latency
- The price of anarchy in this case is $n-1$ for a single commodity and arbitrary latency functions
- Conjecture: The price of anarchy and the Braess ratio are dependent only on the topology, and not the edge latency functions
- Conjecture: The $n-1$ bound is still true for multicommodity networks
- Note that for the average latency objective, there is no separation in the behavior of ρ for single and multicommodity networks

The Braess Ratio in Multicommodity networks

- Perhaps surprisingly, Braess's paradox can be much more severe in multicommodity networks
- There is a phase transition of sorts: While β is polynomial in the single commodity instances, it can be exponential with just two commodities
- Example: A family of networks that is closely related to the Fibonacci numbers. Recall that the p^{th} Fibonacci number is approximately equal to $c \cdot \phi^p$, with $c \simeq 0.4772$ and ϕ the golden ratio

Model

- $G = (V, E)$, K source-destination pairs, traffic rate r_i
- $P_i = s_i - t_i$ paths in G. $\mathcal{P} = \bigcup_{i=1}^{k} P_i$
- Feasible flow f_p routes all traffic. $f_e = \sum_e f_p$
- Latency: $l_p(f) = \sum_{e \in P} l_e(f_e)$, where $l(e)$ is the edge latency function
- Objective: $M(f) = \max_{p \in \mathcal{P}: f_p > 0} l_p(f)$. Let L_i be the common latency of the i^{th} commodity

Theorem (Lin, Roughgarden, Tardos, Walkover 2005)

- There is an infinite family $\{(G^p, r^p, l^p)\}_{p=1}^{\infty}$ with the following properties
- (G^p, r^p, l^p) has two commodities and $O(p)$ vertices and edges;
- For p odd, $L_1(G^p, r^p, l^p) = F_{p-1} + 1$ and $L_2(G^p, r^p, l^p) = F_p$;
- For all p, there is a subgraph H^p of G^p with one less edge than G^p that satisfies $L_1(G^p, r^p, l^p) = 1$ and $L_2(G^p, r^p, l^p) = 0$

Prevalence of Braess's Paradox

- Is Braess's paradox just a mere theoretical curiousity?
- Hardness of finding Braess edges (Roughgarden 2002)
- Theoretical investigations of random graphs show that Braess's paradox does occur with high probability in random graphs as the number of vertices increases
- What is random here?

Model (Valiant and Roughgarden 2006)

- G is the common Erdös-Renyi random graph model
- Each possible edge is present with probability p
- Latency functions are random too, in the following sense: Let the functions be affine (i.e. $ax + b$), where a and b are chosen from some fixed distributions A and B, respectively
- The results are also true for the $1/x$ model: For every edge, the latency is either x (with probability p), or 1 (with probability $1 - p$)

Theorem Valiant and Roughgarden (2006)

- Let A and B be reasonable distributions. There is a constant $\rho = \rho(A, B) > 1$ such that, with high probability, a random network (G, l) admits a choice of traffic rate r such that the Braess ratio of the instance (G, r, l) is at least ρ