

We encourage you to discuss these problems with others, but you need to write up the actual homework alone. At the top of your homework sheet, please list all the people with whom you discussed each problem. Crediting help from other classmates will not take away any credit from you, and will prevent us from assuming cheating if your answers look similar.

1 Thought experiment [10 points]

- Answer the homework question from slide 23 in first lecture. We do not expect anything mathematical – just some intuitive ideas limited to a paragraph or two.

2 Tight or not? [30 points]

In class, we studied a load balancing game where there were n tasks and m servers. One of the settings we looked at in detail was the case where each task could be sent to every machine, i.e., for all tasks i $S_i = S$, and the performance measure was to minimize the maximum load, i.e., for all servers j $r_j(L_j) = L_j$ and $C(A) = \max_j L_j$. In this setting we proved that the price of anarchy was bounded by

$$2 - \frac{2}{m+1}$$

- Determine if this bound is tight. Provide either (a) a family of examples that have price of anarchy matching the bound for all m , or (b) prove that no such family exists, e.g., by proving a tighter bound.

3 Bounded sizes implies bounded PoA [30 points]

For this problem, we will again consider the load balancing game from the previous question. Here, the new wrinkle is the following: the size of the tasks are bounded. In particular, for all i , $p_i \leq \alpha C(A^*)$ where A^* is the assignment that minimizes $C(A)$.

- Prove that the price of anarchy is bounded by $1 + \alpha$.

4 A less explosive game [30 points]

The load balancing games that we considered in class were all *atomic* in the sense that jobs were discrete entities. In this problem, we will consider the case where jobs are not discrete units. Instead, there are job types. This is called the *non-atomic* load balancing game. Our setup is the same as in the atomic case, except that we define x_{ij} as the amount of load that job type i sends to machine j . Thus, an assignment must satisfy

(i) For all job types i , $\sum_{j \in S_i} x_{ij} = p_i$.

(ii) For all j, i , $x_{ij} \geq 0$.

(iii) If $j \notin S_i$, $x_{ij} = 0$.

We also have the following natural extension of the notion of load, $L_j = \sum_i x_{ij}$. We will be considering $C(A) = \max_j r_j(L_j)$.

- (a) Determine all the equilibria in the following simple example. There are two jobs and two servers. Each job can be served by both servers. $p_1 = p_2 = 1$ and $r_1(L) = 2$ while $r_2(L) = L$. Compare the resulting equilibria with those in the atomic load balancing version of this example.
- (b) Write the necessary conditions for a Nash equilibrium in the atomic and non-atomic games.
- (c) (Extra credit) Prove that a Nash equilibrium always exists in the non-atomic game.
- (d) Prove that the price of anarchy is 1 in the non-atomic game. Contrast this with the result for the atomic game.