CS 286a — **HW 1: Practice with orderings** — **Guru: Adam Wierman**

Assigned: 1/19/09 — Due: 1/28/09 10am

I encourage you to discuss these problems with others, but you need to write up the actual solutions alone. At the top of your homework sheet, please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you. Also, please limit your use of the web as it is difficult to develop good problems, and so enough search will probably lead you to find the solutions online.

1 Building intuition for orderings [20 points]

For each pair of random variables, determine which of the following orderings hold: $\leq_{st}$, $\leq_{hr}$, $\leq_{icx}$.

(a) $X \sim Exponential(\mu_X), Y \sim Exponential(\mu_Y)$, $\mu_Y \leq \mu_X$

(b) $X \sim Poisson(\lambda_X), Y \sim Poisson(\lambda_Y)$, $\lambda_X \leq \lambda_Y$

(c) $X \sim Normal(\mu_X, \sigma_X^2), Y \sim Normal(\mu_Y, \sigma_Y^2)$, $\mu_X \leq \mu_Y$ and $\sigma_X^2 = \sigma_Y^2$.

(d) $X \sim Normal(\mu_X, \sigma_X^2), Y \sim Normal(\mu_Y, \sigma_Y^2)$, $\mu_X \leq \mu_Y$ and $\sigma_X^2 \leq \sigma_Y^2$.

(e) $X \sim Deterministic(E[Y])$, Y is an arbitrary distribution.

2 Classes of distributions [45 points]

We have seen a ton of orderings in lecture. There are also a ton of classes of distributions, which in many cases are related to these orderings. Rather than take up class time to study these, we'll develop a feel for them in this problem.

To state some of these definitions, we need the following random variable: X_t has distribution

$$F_t(x) = Pr(X \leq x + t | X > t) = \frac{F(t+x) - F(t)}{\bar{F}(t)}.$$

Definition 1 *A distribution has an* ***increasing (decreasing) failure rate, IFR (DFR)****, if its failure/hazard rate* $\mu_X(t) = f(t)/\bar{F}(t)$ *is non-decreasing (non-increasing) in* t*.*

Definition 2 *A distribution is* ***decreasing (increasing) in mean residual life, DMRL (IMRL)****, if* $E[X_t]$ *is non-increasing (non-decreasing) in* t*.*

Definition 3 *A distribution is* ***new better (worse) than used, NBU (NWU)****, if* $X_t \leq_{st} X$ *for all* t*.*

Definition 4 *A distribution has an* ***increasing (decreasing) likelihood ratio, ILR (DLR)****, if* $X_t \leq_{lr} X_s$ *for* $s < t$*.*

Definition 5 *A distribution is* ***new better (worse) than used in expectation, NBUE (NWUE)****, if* $E[X_t] \leq E[X]$ *for all* t*.*

Now, lets get some practice with these definitions.

(a) Calculate the hazard/failure rate for the Weibull distribution. The $Weibull(\lambda, \alpha)$ distribution is defined as $\bar{F}(t) = e^{-(t/\lambda)^\alpha}$ for $t \geq 0$. Is it IFR or DFR?

(b) Calculate the hazard/failure rate for the uniform distribution. Is it IFR or DFR?

(c) Calculate the hazard/failure rate for the Pareto distribution. The $Pareto(l, \alpha)$ is defined by $\bar{F}(t) = (l/t)^\alpha$ for $t > l$. Is it IFR or DFR?

(d) Prove that IFR is equivalent to $X_t \leq_{st} X_s$ for all $s < t$. Can a stronger ordering be used to relate X_t and X_s?

(e) Prove that IFR is weaker than ILR.

(f) Draw a venn diagram illustrating the relationship between the classes of distributions. For any non-obvious relationships, include justification.

(g) Though NBUE seems like a very weak property, it turns out to be stronger than you might imagine. Recall that the excess distribution is defined as

$$F_{\mathcal{E}(X)} = \int_0^x \frac{\bar{F}_X(t)}{E[X]} dt.$$

Prove that X is NBUE iff $\mathcal{E}(X) \leq_{st} X$. Why is this a useful property when we think about scheduling?

3 Simulating a random variable [35 points]

In class, we discussed a little bit about how to simulate a random variable. In this problem, I'd like you to implement the ideas we discussed (and that were in the handout I mailed to you).

In particular, you should implement a simulator for (i) an Exponential random variable, (ii) a Weibull random variable, and (iii) a Pareto random variable.

You should turn in (1) a copy of your code, and (2) histograms illustrating a sequence of 10, 100, 1000 samples from each of

(a) Exponential with mean 1

(b) Pareto with mean 1 and $\alpha = 1.2$.

(c) Pareto with mean 1 and $\alpha = 4$.

(d) Weibull with mean 1 and $\alpha = 0.5$.

(e) Weibull with mean 1 and $\alpha = 2$.

For each sequence of histograms, summarize the mean and variance of the distributions. Also, discuss the results: which distributions converge quickly/slowly and why?

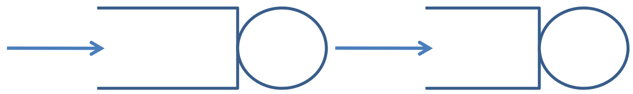

Figure 1: A tandem queue.

4 Can you couple this tandem? [15 points extra credit]

Prove, using coupling, that a GI/GI/1-/GI/1 tandem queue (see Figure 1) with $E[A] > E[X_1]$ and $E[A] > E[X_2]$ converges to a stationary distribution, where X_i follows the job size distribution at station i and A is an interarrival time.

Note: You may assume, as we did in class, that a queue with negative drift will empty a.s. in a finite time.
Hint: Start by focusing on just the first queue.
Hint: The mean interarrival time at both queues is the same – why is this?