An easy-to-memorize version of master theorem

The master theorem (Theorem 4.1 in CLRS) is concerned with recurrences of the form $T(n) = aT(n/b) + f(n)$. It consists of three cases: (1) $f(n) = O(n^\alpha)$, (2) $f(n) = \Theta(n^\alpha)$, and (3) $f(n) = \Omega(n^\alpha)$, where α is some constant. We will only consider the second case; the other two are easy to derive from it. (This way one gets a slightly weaker statement in case (3), but that is not important.)

Theorem 1. *Let $a \geq 1,\ b > 1,\ \alpha \geq 0$ be some constants, and let $T(n)$ satisfy the recurrence*[1]

$$T(n) = a\,T(\underbrace{q(n)}_{e.g.,\ \lceil n/b \rceil}) + \Theta(n^\alpha), \qquad \textit{where} \quad |q(n) - n/b| \leq O(1). \tag{1}$$

Then

$$T(n) = \begin{cases} \Theta(n^\beta), & \textit{if } \alpha < \beta, \\ \Theta(n^\beta \log n), & \textit{if } \alpha = \beta, \\ \Theta(n^\alpha), & \textit{if } \alpha > \beta, \end{cases} \qquad \textit{where} \quad \beta = \log_b a. \tag{2}$$

Informal proof. Let's ignore the $O(1)$ term in Eq. (1) and replace $\Theta(n^\alpha)$ by cn^α. Then

$$T(n) = cn^\alpha + aT(n/b) = cn^\alpha + a\big(cn^\alpha + aT(n/b^2)\big) = cn^\alpha + a\underbrace{\Big(c(n/b)^\alpha + a\,\underbrace{\big(c(n/b^2)^\alpha + \ldots\big)}_{T(n/b^2)}\Big)}_{T(n/b)}$$

$$= cn^\alpha\Big(1 + \big(a/b^\alpha\big) + \big(a/b^\alpha\big)^2 + \cdots + \big(a/b^\alpha\big)^t\Big) = \begin{cases} \Theta(n^\alpha), & \text{if } a/b^\alpha < 1, \\ \Theta(n^\alpha t), & \text{if } a/b^\alpha = 1, \\ \Theta\big(n^\alpha (a/b^\alpha)^t\big), & \text{if } a/b^\alpha > 1, \end{cases}$$

where $t = \log_b n$. Notice that $a/b^\alpha = b^{\beta-\alpha}$, therefore $n^\alpha (a/b^\alpha)^t = n^\beta$. □

Proof. An estimate of the form $T(n) = g(n)$ is equivalent to two inequalities: $T(n) \leq O(g(n))$ (upper bound) and $T(n) \geq \Omega(g(n))$ (lower bound). We will only proof the upper bound for $\alpha < \beta$. For this purpose, we may weaken the assumption of the theorem, replacing the equality by an inequality. The exact statement we are going to prove is this:

Let $a \geq 1,\ b > 1,\ d \geq 0$, and $0 < \alpha < \beta \stackrel{\text{def}}{=} \log_b a$ be some constants, and let

$$T(n) \leq a\,T(q(n)) + f(n), \qquad \textit{where} \quad f(n) = O(n^\alpha), \quad 0 \leq q(n) \leq n/b + d. \tag{3}$$

Then

$$T(n) \leq O(n^\beta). \tag{4}$$

The other 5 cases are analyzed similarly.

The proof method is simple: we replace $O(n^\beta)$ in inequality (3) by a concrete function $h(n)$ and prove the result by induction. However, this may or may not work depending on the choice

[1] Since the recurrence contains $\Theta(\ldots)$, it is only meaningful for sufficiently large n, say $n \geq n_0$, whereas for smaller n the function $T(n)$ is completely arbitrary.

of h. (In such situations, one often uses trial and error, but a good strategy helps a lot.) Let us first reduce the problem to the special case where $d = 0$. This is achieved by a variable change, $n \mapsto \tilde{n} = n - p$ for a suitable constant p. Specifically, let

$$\begin{aligned} p &= \lceil db/(b-1) \rceil, & \tilde{q}(\tilde{n}) &= q(\tilde{n}+p) - p, \\ \tilde{T}(\tilde{n}) &= T(\tilde{n}+p), & \tilde{f}(\tilde{n}) &= f(\tilde{n}+p), \end{aligned} \tag{5}$$

so that Eq.(3) becomes $\tilde{T}(\tilde{n}) \le a\tilde{T}(\tilde{q}(\tilde{n})) + \tilde{f}(\tilde{n}^\alpha)$. It is easy to see that $-p \le \tilde{q}(\tilde{n}) \le \tilde{n}/b$, $\tilde{f}(\tilde{n}) = O(\tilde{n}^\alpha)$, so we have obtained the same kind of problem with a slightly different condition on q. We will keep that difference in mind, but drop tildes from the notation.

Now let us write the theorem assumption with explicit constants instead of $O(\ldots)$ and formulate an exact bound on $T(n)$:

Let $a \ge 1$, $b > 1$, $c, d, p, n_0 > 0$, and $0 < \alpha < \beta \stackrel{\text{def}}{=} \log_b a$ be some constants, and let

$$T(n) \le a\,T(q(n)) + cn^\alpha \quad \text{for all } n \ge n_0, \qquad \text{where} \quad -p \le q(n) \le n/b. \tag{6}$$

Then there are some numbers $u > 0$ and v such that the function $h(n) = un^\beta + vn^\alpha$ is monotone for $n \ge n_0$ and

$$T(n) \le \max\{h(n), h(n_0)\} \quad \text{for all } n \ge -p. \tag{7}$$

The parameters u and v will be tuned later so as to make the induction work. Note that v need not be positive, and we will actually assign it a negative value!

Base case: *Inequality (7) holds for all $n < bn_0$.* That is certainly true if

$$h(n_0) \le \max_{n \le bn_0} T(n). \tag{8}$$

We will satisfy inequality (8) by tuning the parameters.

Inductive step: *Let $n \ge bn_0$. If $T(m) \le h(m)$ for all $m < n$, then $T(n) \le h(n)$.* We are trying to prove that.

$$T(n) \underset{\substack{\text{inequality}\\(6)}}{\le} a\,T(q(n)) + cn^\gamma \underset{\substack{\text{induction}\\\text{hypothesis}}}{\le} a\,\max\{h(q(n)), h(n_0)\} + cn^\alpha \underset{\text{monotonicity}}{\le} a\,h(n/b) + cn^\alpha$$

$$= (a/b^\beta)un^\beta + (a/b^\alpha)vn^\alpha + cn^\alpha = un^\beta + (sv+c)n^\alpha,$$

where $s = a/b^\alpha > 1$. The last expression in the above calculation is bounded by $h(n) = un^\beta + vn^\alpha$, provided

$$(s-1)v + c \le 0. \tag{9}$$

Now we can tune the parameters u and v. We need to satisfy inequalities (8) and (9), as well as the monotonicity condition:

$$h'(x) = u\beta x^{\beta-1} + v\alpha x^{\alpha-1} \ge 0 \quad \text{for } x \ge n_0. \tag{10}$$

Here is one possible solution:

$$v = -c/(s-1), \qquad u = -vn_0^{\alpha-\beta} + n_0^{-\beta} \max\left\{\max_{n \le bn_0} T(n),\, 0\right\}. \tag{11}$$

□

Theorem 1 and its inductive proof (but not the nice "intuitive proof") can easily be generalized to recurrences of the form $T(n) = \sum_{j=1}^{k} a_j T(n/b_j) + \Theta(n^\alpha)$. All we need is to define β as the solution to this equation:

$$\sum_{j=1}^{k} a_j / b_j^\beta = 1. \tag{12}$$

Thus we get this result:

Theorem 2. *Let $a_j \geq 1$, $b_j > 1$ $(j = 1, \ldots, k)$, and $\alpha \geq 0$ be some constants, and let $T(n)$ satisfy the recurrence*

$$T(n) = \sum_{j=1}^{k} a_j T(q_j(n)) + \Theta(n^\alpha), \qquad \textit{where} \quad |q_j(n) - n/b_j| \leq O(1). \tag{13}$$

Then

$$T(n) = \begin{cases} \Theta(n^\beta), & \textit{if } \alpha < \beta, \\ \Theta(n^\beta \log n), & \textit{if } \alpha = \beta, \\ \Theta(n^\alpha), & \textit{if } \alpha > \beta, \end{cases}, \qquad \textit{where } \beta \textit{ is the solution of equation (12).} \tag{14}$$