The derivative of a determinant

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Abstract? No, not really. This is a *classical* result.

Background and a simple result

Let $\Phi(t)$ be an $n \times n$ matrix depending on a parameter t. If Φ is a differentiable function of t — that is, each of its components is differentiable with respect to t — then so is $\det \Phi(t)$, since we know that the determinant is a polynomial in the components of Φ. To get from this to an actual computation of the derivative of $\det \Phi(t)$ is a different matter, though.

What we shall need is the fact that *the determinant is a multilinear function of its rows*: If we write the rows of Φ as $\varphi_1, \dots, \varphi_n$ and think of the determinant as a function of the rows

$$\det \Phi = d(\varphi_1, \dots, \varphi_n)$$

then d is a linear function of each of its arguments as long as we keep each of the remaining rows constant. We then get

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \Phi(t) = d(\dot\varphi_1, \varphi_2, \dots, \varphi_n) + d(\varphi_1, \dot\varphi_2, \dots, \varphi_n) + \cdots + d(\varphi_1, \varphi_2, \dots, \dot\varphi_n) \qquad (1)$$

I outline the proof of this only for $n = 3$, to keep the notation simple. It should be clear how to generalize the proof to arbitrary n. If $h \neq 0$ then

$$\begin{aligned} h^{-1}\big(\Phi(t+h) - \Phi(t)\big) &= d\big(h^{-1}(\varphi_1(t+h) - \varphi_1(t)), \varphi_2(t+h), \varphi_3(t+h)\big) \\ &+ d\big(\varphi_1(t), h^{-1}(\varphi_2(t+h) - \varphi_2(t)), \varphi_3(t+h)\big) \\ &+ d\big(\varphi_1(t), \varphi_2(t), h^{-1}(\varphi_3(t+h) - \varphi_3(t))\big) \end{aligned}$$

which has the stated limit as $h \to 0$. (We must use the continuity of d for this argument to work.)

A better result

Equation (1) requires the computation of n determinants for the computation of a single derivative. We can do much better than this! For example, if $\Phi(t)$ is the identity matrix then a moment's contemplation of the righthand side of (1) shows it is the trace of $\dot\Phi$. Indeed, the first term will be

$$\begin{vmatrix} \dot\varphi_{11} & \dot\varphi_{12} & \cdots & \dot\varphi_{1n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = \dot\varphi_{11}$$

and so forth, with the sum $\dot\varphi_{11} + \dot\varphi_{22} + \cdots + \dot\varphi_{nn} = \operatorname{tr} \dot\Phi$. The resulting formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \Phi(t) = \operatorname{tr} \dot\Phi(t) \quad \text{when } \Phi(t) = I$$

may seem like a rather useless special case, but appearances deceive! For, let A be a constant, invertible matrix and apply the above result to the function $\det(A\Phi(t)) = \det A \, \det \Phi(t)$. Now, the above formula states that

$$\det A \frac{\mathrm{d}}{\mathrm{d}t} \det \Phi(t) = \operatorname{tr}(A\dot\Phi(t)) \quad \text{when } A\Phi(t) = I$$

Whenever $\Phi(t)$ is invertible we can apply this result with $A = \Phi(t)^{-1}$ and rearrange to get the result

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \Phi(t) = \det \Phi(t) \; \operatorname{tr} \left(\Phi(t)^{-1} \dot\Phi(t)\right)$$

This result can also be written in the following useful form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \det \Phi(t) = \operatorname{tr} \left(\Phi(t)^{-1} \dot\Phi(t)\right).$$