Ph/CS 219, Fall 2009
Quantum Computation
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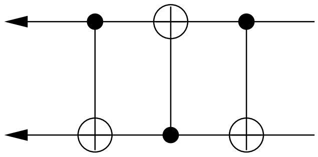
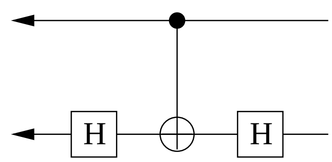
Solution set #1

1. *(10 points) Find the operators represented by the following circuits:*

a) $W_1 =$  ; *b)* $W_2 =$  .

Write these operators as matrices in the standard basis. Describe what the first operator does (it's really simple!) Represent the second operator as

$$\Lambda(U) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U \tag{1}$$

for a suitable U acting on the second qubit. Do W_1 and W_2 commute?

We can do this question in two ways: (i) by following the evolution of the basis states, or (ii) by multiplying matrices. We'll calculate W_1 the the first way and use a mixed approach for W_2.

(a) The basis states evolve as follows:

$$|a, b\rangle \xrightarrow{\text{CNOT}[1,2]} |a,\ b \oplus a\rangle \xrightarrow{\text{CNOT}[2,1]} |b,\ b \oplus a\rangle \xrightarrow{\text{CNOT}[1,2]} |b, a\rangle. \tag{2}$$

Thus, W_1 simply swaps the two qubits. We can also represent it by a matrix:

$$W_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

(b) The gates in this circuit do not change the classical value a of the first qubit. Thus, W_2 can be written in the form

$$W_2 = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1 = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix}, \tag{4}$$

where U_a $(a = 0, 1)$ acts on the second qubit. Specifically, $U_0 = HIH = I$ and $U_1 = H\sigma^x H = \sigma^z$. We have conclude that

$$W_2 = \Lambda(\sigma^z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5}$$

It is easy to check that W_1 and W_2 commute.

2. *(10 points) Suppose we can prepare qubits in the state* $|0\rangle$ *and act on them by the gates* H, σ^x, σ^y, σ^z, *and* CNOT. *It's clear that this set of operations is insufficient to create the state*

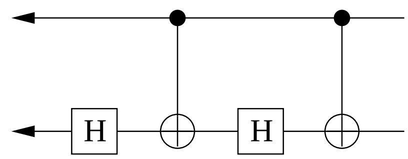
$$|\eta_+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + i|1\rangle\Big) \quad \textit{or} \quad |\eta_-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - i|1\rangle\Big), \tag{6}$$

even up to an overall phase factor. Indeed, the operations we use have real coefficients. (Well, almost: $\sigma^y = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *but we don't care about the overall factor.)*

Show that it is, however, possible to copy an unknown state $|\psi\rangle$ *with respect to the basis* $\big\{|\eta_+\rangle, |\eta_-\rangle\big\}$. ***Hint:*** *Using the above gate set, construct a circuit that performs* $\Lambda(i\sigma^y)$, *i.e., the controlled* $i\sigma^y$. *Prepare a suitable state in the first (controlling) qubit and send* $|\psi\rangle$ *to the second (controlled) qubit.*

Let us first implement $\Lambda(i\sigma^y)$. Since $i\sigma^y = \sigma^z\sigma^x$, it follows that

$$\Lambda(i\sigma^y) = \Lambda(\sigma^z)\Lambda(\sigma^x) = \tag{7}$$



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see solution to problem 1b. Now, we apply this operator to the product of states $|+\rangle = H|0\rangle$ (which we can construct) and $|\eta_\pm\rangle$ (which we want to copy). Notice that $\sigma^y|\eta_\pm\rangle = \pm|\eta_\pm\rangle$. When we act by $\Lambda(i\sigma^y)$, the $\pm i$ factor is applied selectively, depending on the classical value of the first qubit:

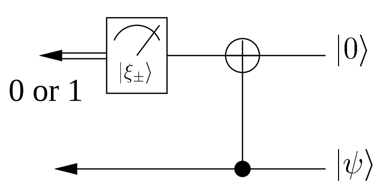
$$\Lambda(i\sigma^y)\left(\frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big) \otimes |\eta_\pm\rangle\right) = \frac{1}{\sqrt{2}}\Big(|0\rangle \pm i|1\rangle\Big) \otimes |\eta_\pm\rangle = |\eta_\pm\rangle \otimes |\eta_\pm\rangle \tag{8}$$

(all the $\pm$ signs are the same).

3. Computation by measurement. *(10 points) Suppose we can measure an arbitrary state with respect to this basis:*

$$|\xi_+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\varphi}|1\rangle\Big), \qquad |\xi_-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - e^{i\varphi}|1\rangle\Big). \tag{9}$$

Consider the following circuit acting on a pure state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$*:*



a) Write the output density matrix in the form $\rho = \begin{pmatrix} \rho_0 & 0 \\ 0 & \rho_1 \end{pmatrix}$, *where* ρ_0 *and* ρ_1 *are* non-normalized *density matrices of the second qubit corresponding to the two measurement outcomes (*$\operatorname{Tr}\rho = \operatorname{Tr}\rho_0 + \operatorname{Tr}\rho_1 = 1$*).*

b) *Show that* $\operatorname{Tr}\rho_0 = \operatorname{Tr}\rho_1 = 1/2$ *(which implies that each outcome occurs with probability* $1/2$ *regardless of the input state). Now, define the normalized* conditional states $\tilde{\rho}_x = 2\rho_x$ *(*$x = 0, 1$*) and interpret them as the result of application of some unitaries* U_0*,* U_1 *to* $|\psi\rangle$*. (Note that such an interpretation is not always possible since normalizing a state is generally a nonlinear operation. But in this case, we are lucky.)*

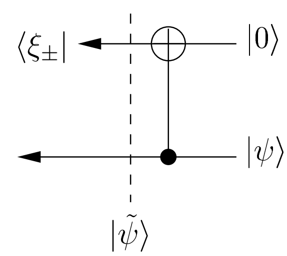
c) *Explain how to use this circuit together with a classically controlled* σ^z *to implement the unitary gate*

$$\Lambda\big(e^{-i\varphi}\big) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \tag{10}$$

*(*Classically controlled *means that maintaining coherence between the control bit states is not necessary. In our case, we use the measurement outcome as the control.)*

a) The two-qubit state before the measurement is $\gamma = |\tilde{\psi}\rangle\langle\tilde{\psi}|$, where $|\tilde{\psi}\rangle = c_0|00\rangle + c_1|11\rangle$. The measurement is represented by the projectors $\Pi_0 = |\xi_+\rangle\langle\xi_+|$ and $\Pi_1 = |\xi_-\rangle\langle\xi_-|$. If there were only one qubit, we would simply compute the numbers $p_a = \operatorname{Tr}(\Pi_a\gamma)$ $(a = 0, 1)$. But since Π_a only acts on one of the two qubits, the numbers are replaced by unnormalized density matrices:

$$\rho_a = \operatorname{Tr}_1\big((\Pi_a \otimes I)\gamma\big) = |\mu_a\rangle\langle\mu_a|, \quad \text{where } |\mu_a\rangle = \big(\langle\xi_\pm| \otimes I\big)|\tilde{\psi}\rangle =$$



(11)

where $a = 0, 1$ corresponds to $|\xi_+\rangle$ and $|\xi_-\rangle$, respectively. The expression $\big(\langle\xi_\pm| \otimes I\big)|\tilde{\psi}\rangle$ or its graphical equivalent looks a bit confusing, so here is the exact recipe: If $|\xi\rangle = \sum_a u_a|a\rangle$ is some single-qubit state and $|\eta\rangle = \sum_{a,b} v_{ab}|ab\rangle$ is a two-qubit state, then

$$\big(\langle\xi| \otimes I\big)|\tilde{\eta}\rangle = \sum_b \left(\sum_a u_a^* v_{ab}\right)|b\rangle. \tag{12}$$

In our case, we get

$$|\mu_0\rangle = \frac{1}{\sqrt{2}}\Big(c_0|0\rangle + e^{-i\varphi}c_1|1\rangle\Big), \qquad |\mu_1\rangle = \frac{1}{\sqrt{2}}\Big(c_0|0\rangle - e^{-i\varphi}c_1|1\rangle\Big). \tag{13}$$

b) We readily see that $|\mu_a\rangle = \frac{1}{\sqrt{2}}U_a|\psi\rangle$, where

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -e^{-i\varphi} \end{pmatrix}. \tag{14}$$

The square of the normalization factor $\frac{1}{\sqrt{2}}$ gives the measurement outcome probability, whereas U_0 and U_1 are unitary.

c) The previous result implies that if we got outcome a, the qubit has undergone the transformation U_a. If $a = 0$, we are done. If $a = 1$, we need to apply σ^z to fix the sign. Thus, the application of σ^z is controlled by the (classical) measurement outcome.

4. ***Positivity vs. complete positivity*** *(10 points) Let $T : \mathbf{L}(\mathbb{C}^2) \to \mathbf{L}(\mathbb{C}^2)$ be a superoperator defined by the equations*

$$TI = I, \quad T\sigma_x = x\sigma_x, \quad T\sigma_y = y\sigma_y, \quad T\sigma_z = z\sigma_z, \tag{15}$$

where x, y, z are some real numbers.

a) Find a necessary and sufficient condition for T being positive.

b) Find a necessary and sufficient condition for T being completely positive. ***Hint:*** *Use the matrix representation.*

a) Using the Bloch sphere picture, we can write an arbitrary matrix in the form

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \tag{16}$$

with $|\vec{r}| \leq 1$. It follows that

$$T\rho = \frac{1}{2}(I + r_x x\sigma_x + r_y y\sigma_y + r_z z\sigma_z), \tag{17}$$

which is a valid density matrix provided $r_x^2 x^2 + r_y^2 y^2 + r_z^2 z^2$. Taking $\vec{r} = (1, 0, 0)$ implies that $|x| \leq 1$; similarly we must have $|y| \leq 1$ and $|z| \leq 1$. These conditions are also sufficient because they imply $r_x^2 x^2 + r_y^2 y^2 + r_z^2 z^2 \leq r_x^2 + r_y^2 + r_z^2 \leq 1$. Thus, T is positive iff $|x|, |y|, |z| \leq 1$.

b) We write T in the matrix representation as

$$T\big(|j\rangle\langle k|\big) = \sum_{j',k'} \breve{T}_{(j'j)(k'k)} |j'\rangle\langle k'|. \tag{18}$$

The matrix elements $\breve{T}_{(j',j)(k',k)}$ are defined by Eq. (15). In particular,

$$T\big(|0\rangle\langle 0|\big) = T\left(\frac{I + \sigma^z}{2}\right) = \frac{I + z\sigma^z}{2} = \frac{1+z}{2}|0\rangle\langle 0| + \frac{1-z}{2}|1\rangle\langle 1|, \tag{19}$$

$$T\big(|0\rangle\langle 1|\big) = T\left(\frac{\sigma^x + i\sigma^y}{2}\right) = \frac{x\sigma^x + iy\sigma^y}{2} = \frac{x+y}{2}|0\rangle\langle 1| + \frac{x-y}{2}|1\rangle\langle 0|, \tag{20}$$

$$T\big(|1\rangle\langle 0|\big) = \frac{x-y}{2}|0\rangle\langle 1| + \frac{x+y}{2}|1\rangle\langle 0|, \tag{21}$$

$$T\big(|1\rangle\langle 1|\big) = \frac{1-z}{2}|0\rangle\langle 0| + \frac{1+z}{2}|1\rangle\langle 1|. \tag{22}$$

Thus,

$$\breve{T} = \frac{1}{2}\begin{pmatrix} 1+z & 0 & 0 & x+y \\ 0 & 1-z & x-y & 0 \\ 0 & x-y & 1-z & 0 \\ x+y & 0 & 0 & 1+z \end{pmatrix} \tag{23}$$

As discussed in class, a superoperator T is completely positive iff it is physically realizable, which is equivalent to these three conditions:

1. The matrix $\breve{T} = (\breve{T}_{JK})$ is Hermitian;
2. The matrix $\breve{T}$ is positive;
3. $\sum_s \breve{T}_{(sj)(sk)} = \delta_{jk}$.

It is clear that conditions 1 and 3 are satisfied; we must find for which values of x, y, and z condition 2 holds. Finding the eigenvalues of $\breve{T}$ gives four inequalities:

a) $x + y - z \leq 1$;

b) $x - y + z \leq 1$;

c) $-x + y + z \leq 1$;

d) $-x - y - z \leq 1$.

These are necessary and sufficient for T being completely positive. Geometrically, this set of inequalities defines a tetrahedron with vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$.