Ph/CS 219, Winter 2005
Quantum Computation
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Problem set #2
Due Monday, February 14, 2005

1. Logical gates on the five-qubit code (15 points) Recall that the five-qubit code is a $[[5,1,3]]$ symplectic code given by this set of check operators:

$$\begin{aligned} S_1 &= X\ Z\ Z\ X\ I \\ S_2 &= Z\ Z\ X\ I\ X \\ S_3 &= Z\ X\ I\ X\ Z \\ S_4 &= X\ \ I\ X\ Z\ Z. \end{aligned} \tag{1}$$

a) Show that the operators

$$\hat{X} \;=\; XXXXX \quad \text{and} \quad \hat{Z} \;=\; ZZZZZ \tag{2}$$

may be used as logical σ^x and σ^z. Specifically, you need to check two things: (i) these operators preserve the code subspace $\mathcal{M}$; (ii) they obey the same algebraic relations as σ^x and σ^z, namely,

$$\hat{X}^\dagger = \hat{X}, \quad \hat{Z}^\dagger = \hat{Z}, \quad \hat{X}^2 = \hat{Z}^2 = 1, \qquad \hat{X}\hat{Z} = -\hat{Z}\hat{X}. \tag{3}$$

What is logical σ^y? Also show that the logical $|0\rangle$ and $|1\rangle$ are given by this formula:

$$\begin{aligned} |0^L\rangle &= 4\Pi|00000\rangle \\ |1^L\rangle &= 4\Pi|11111\rangle, \end{aligned} \qquad \text{where} \quad \Pi = \frac{(1+S_1)(1+S_2)(1+S_3)(1+S_4)}{2^4}. \tag{4}$$

b) Consider the one-qubit operator

$$T \;=\; e^{i\pi/4}KH \;=\; \frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{5}$$

Note that T belong to the Cifford group and acts on the Pauli operators as follows:

$$T\sigma^x T^\dagger = \sigma^z, \quad T\sigma^z T^\dagger = \sigma^y, \quad T\sigma^y T^\dagger = \sigma^x. \tag{6}$$

Show that the operator $\hat{T} = T^{\otimes 5}$ preserves the code subspace and may be interpreted as the logical T up to a phase factor. **Hint:** Find the action of $\hat{T}$ on $\hat{X}$ and $\hat{Z}$. By computing the eigenvalues of $\hat{T}$, one can prove that the unknown phase is trivial (see below).

c) Let $|\psi_+\rangle$ and $|\psi_-\rangle$ be eigenvectors of T corresponding to the eigenvalues $e^{+i\pi/3}$ and $e^{-i\pi/3}$, respectively. Compute $\langle\psi_+^{\otimes 5}|\Pi|\psi_+^{\otimes 5}\rangle$ and $\langle\psi_-^{\otimes 5}|\Pi|\psi_-^{\otimes 5}\rangle$. **Hint:** Calculate all 16 terms in the expansion of $(1+S_1)(1+S_2)(1+S_3)(1+S_4)$. (You may save some time by using the cyclic symmetry of the code.) Then use the formula

$$|\psi_\pm\rangle\langle\psi_\pm| \;=\; \frac{1}{2}\left(I \pm \frac{1}{\sqrt{3}}(\sigma^x+\sigma^y+\sigma^z)\right). \tag{7}$$

d) Show that

$$T^{\otimes 5}\Pi|\psi_+^{\otimes 5}\rangle = e^{-i\pi/3}|\psi_+^{\otimes 5}\rangle, \qquad T^{\otimes 5}\Pi|\psi_-^{\otimes 5}\rangle = e^{+i\pi/3}|\psi_-^{\otimes 5}\rangle. \tag{8}$$

2. Symplectic states (15 points) A state $|\psi\rangle$ is called *symplectic* if it is the unique (up to a phase) codevector of a symplectic code of dimension 1. In other words, $|\psi\rangle$ is defined by the condition

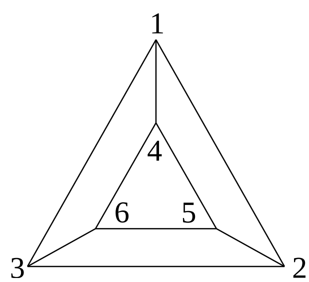
$$\sigma(f_j)|\psi\rangle = (-1)^{\mu_j}|\psi\rangle, \qquad f_j \in \mathbb{F}_2^{2n}, \quad \mu_j \in \{0,1\} \quad \text{for} \quad j = 1, \ldots, n, \tag{9}$$

where f_j are linearly independent and $\omega(f_j, f_k) = 0$ (i.e., f_j span a Lagrangian subspace).[1]

a) Consider the following state of six qubits:[2]

$$|\psi\rangle \;=\; 2^{-3} \sum_{x_1,\ldots,x_6} (-1)^{\sum_{(j,k)\in E} x_j x_k} |x_1, \ldots, x_6\rangle, \tag{10}$$

where E is the set of edges of this graph:



Represent $|\psi\rangle$ as a common eigenvector of six independent Pauli operators $S_1, \ldots, S_6$. **Hint:** look for operators that are products of one σ^x and several σ^z.

b) Show that the state (10) is perfectly 3-entangled. (A pure n-qubit state is called *perfectly k-entangled* if the reduced density matrix on any k qubits is equal to $\left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right)^{\otimes k}$.)

c) By removing qubit 6 from the state (10) we obtain a $[[5,1]]$ code C. (It actually coincides with the standard five-qubit code up to a local basis change.) Write a set of check operators for the code C. **Hint:** Find products of $S_1, \ldots, S_6$ that do not involve qubit 6.

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[1] We use the same notation as in class:

$$\sigma(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) \stackrel{\text{def}}{=} \sigma^{\alpha_1\beta_1} \otimes \cdots \otimes \sigma^{\alpha_n\beta_n}, \qquad \sigma^{00} = I, \quad \sigma^{01} = \sigma^z, \quad \sigma^{10} = \sigma^x, \quad \sigma^{11} = \sigma^y.$$

$$\omega(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n;\, \alpha_1', \beta_1', \ldots, \alpha_n', \beta_n') \stackrel{\text{def}}{=} \sum_{j=1}^{n} \alpha_j \beta_j' - \beta_j \alpha_j' \pmod 2.$$

[2] This state was studied by A. Calderbank, E. Rains, P. Shor, and N. Sloane, `quant-ph/9608006` and by S. Bravyi, `quant-ph/0205021`.

3. General relation between codes and entangled states (15 points)

a) Let $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ be the Hilbert space of n qubits, $\mathcal{L}$ an auxiliary space, and $|\psi\rangle \in \mathcal{H} \otimes \mathcal{L}$ a pure state with the Schmidt decomposition

$$|\psi\rangle = \sum_{j=1}^{m} \lambda_j |\xi_j\rangle \otimes |\eta_j\rangle, \qquad |\xi_j\rangle \in \mathcal{H}, \quad |\eta_j\rangle \in \mathcal{L}, \quad \lambda_j > 0. \tag{11}$$

Let us define a code subspace $\mathcal{M}$ as the linear span of $|\xi_1\rangle, \ldots, |\xi_m\rangle$. Prove that $\mathcal{M}$ detects k errors if and only if for any subset A of k qubits the reduced density matrix $\rho_{A\mathcal{L}} = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$ (where B stands for the remaining qubits) has the form

$$\rho_{A\mathcal{L}} = \rho_A \otimes \rho_{\mathcal{L}}. \tag{12}$$

(We have used this property in class. I also gave a proof of an equivalent statement; you just need to put all that together.)

b) Prove that a stabilizer state is perfectly k-entangled if and only if any product of one or more check operators acts nontrivially on more than k qubits. **Hint:** The latter condition can be rephrased as follows: any nonzero vector from the associated isotropic subspace F (the linear span of f_j) has magnitude greater than k. (The magnitude of a vector $g = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$ is the number of nonzero pairs (α_j, β_j).) The perfect entanglement property is actually related to the space of errors with trivial syndrome,

$$F_+ = \left\{ g \in \mathbb{F}_2^{2n} : \ \forall j \ \omega(f_j, g) = 0 \right\} \tag{13}$$

For general stabilizer codes, $F \subseteq F_+$. But for stabilizer states, $F = F_+$.