Ph/CS 219, Fall 2009
Quantum Computation
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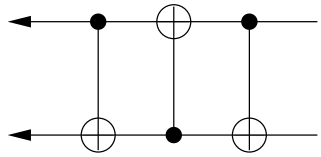
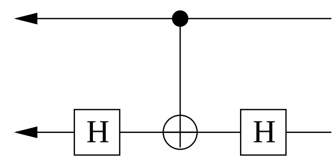
Problem set #1
Due Monday, October 19, 2009

1. (10 points) Find the operators represented by the following circuits:

a) $W_1 =$  ; b) $W_2 =$ 

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Write these operators as matrices in the standard basis. Describe what the first operator does (it's really simple!) Represent the second operator as

$$\Lambda(U) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U \tag{1}$$

for a suitable U acting on the second qubit. Do W_1 and W_2 commute?

2. (10 points) Suppose we can prepare qubits in the state $|0\rangle$ and act on them by the gates H, σ^x, σ^y, σ^z, and CNOT. It's clear that this set of operations is insufficient to create the state

$$|\eta_+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + i|1\rangle\Big) \quad \text{or} \quad |\eta_-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - i|1\rangle\Big), \tag{2}$$

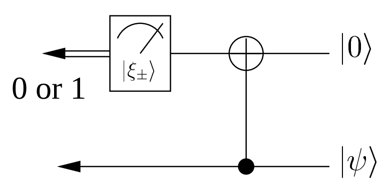
even up to an overall phase factor. Indeed, the operations we use have real coefficients. (Well, almost: $\sigma^y = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ but we don't care about the overall factor.)

Show that it is, however, possible to copy an unknown state $|\psi\rangle$ with respect to the basis $\big\{|\eta_+\rangle, |\eta_-\rangle\big\}$. **Hint:** Using the above gate set, construct a circuit that performs $\Lambda(i\sigma^y)$, i.e., the controlled $i\sigma^y$. Prepare a suitable state in the first (coltrolling) qubit and send $|\psi\rangle$ to the second (controlled) qubit.

3. Computation by measurement. (10 points) Suppose we can measure an arbitrary state with respect to this basis:

$$|\xi_+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\varphi}|1\rangle\Big), \qquad |\xi_-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - e^{i\varphi}|1\rangle\Big). \tag{3}$$

Consider the following circuit acting on a pure state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$:



a) Write the output density matrix in the form $\rho = \begin{pmatrix} \rho_0 & 0 \\ 0 & \rho_1 \end{pmatrix}$, where ρ_0 and ρ_1 are *non-normalized* density matrices of the second qubit corresponding to the two measurement outcomes ($\operatorname{Tr}\rho = \operatorname{Tr}\rho_0 + \operatorname{Tr}\rho_1 = 1$).

b) Show that $\operatorname{Tr}\rho_0 = \operatorname{Tr}\rho_1 = 1/2$ (which implies that each outcome occurs with probability $1/2$ regardless of the input state). Now, define the normalized *conditional states* $\tilde{\rho}_x = 2\rho_x$ ($x = 0, 1$) and interpret them as the result of application of some unitaries U_0, U_1 to $|\psi\rangle$. (Note that such an interpretation is not always possible since normalizing a state is generally a nonlinear operation. But in this case, we are lucky.)

c) Explain how to use this circuit together with a classically controlled σ^z to implement the unitary gate

$$\Lambda\left(e^{-i\varphi}\right) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \tag{4}$$

(*Classically controlled* means that maintaining coherence between the control bit states is not necessary. In our case, we use the measurement outcome as the control.)

4. Positivity vs. complete positivity (10 points) Let $T : \mathbf{L}(\mathbb{C}^2) \to \mathbf{L}(\mathbb{C}^2)$ be a superoperator defined by the equations

$$TI = I, \qquad T\sigma_x = x\sigma_x, \qquad T\sigma_y = y\sigma_y, \qquad T\sigma_z = z\sigma_z, \tag{5}$$

where x, y, z are some real numbers.

a) Find a necessary and sufficient condition for T being positive.

b) Find a necessary and sufficient condition for T being completely positive. **Hint:** Use the matrix representation.