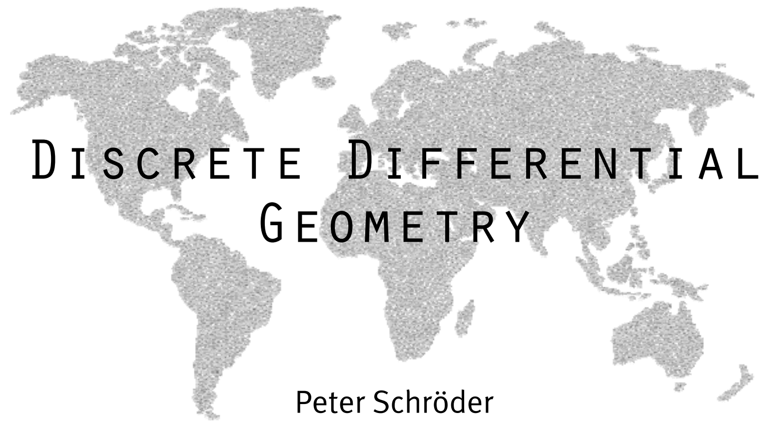

Discrete Differential Geometry

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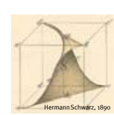
A Bit of History

Geometry is the key!

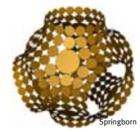


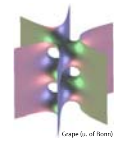
- studied for centuries
  - Cartan, Poincaré, Lie, Hodge, de Rham, Gauss, Noether,...
- mostly differential geometry
  - differential and integral calculus

Study of invariants and symmetries

Differential Geometry

Why do we care?

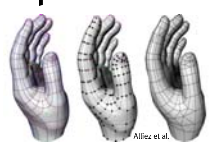


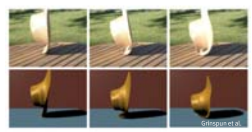


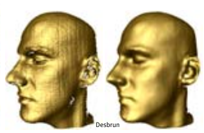
- geometry of surfaces
- mothertongue of physical theories

$$E = \int_S \alpha + \beta(H - H_0)^2 + \gamma K \, dA$$

- computation: simulation/processing







Getting Started

How to apply DiffGeo ideas?

- surfaces as collections of samples
  - and topology (connectivity)
- apply continuous ideas
  - BUT: setting is discrete
- what is the right way?
  - discrete vs. discretized

Discretized

Build smooth manifold structure

- collection of charts
  - mutually compatible on their overlaps
- form an atlas
- realize as smooth functions
  - differentiate away...

Discretization of Eqs

Observation

- be careful

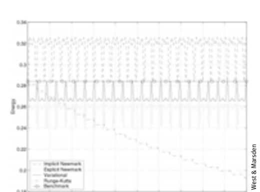
$$M\ddot{q}+C\dot{q}+Kq=f$$

?

$$m\frac{q_{i-1}-2q_i+q_{i+1}}{\Delta t^2}+c\frac{q_{i+1}-q_i}{\Delta t}+kq_{i+1}=f_i$$

- structure may not be preserved

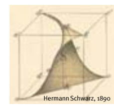
$$L(q,\dot{q})=\frac{1}{2}\dot{q}M\dot{q}-V(q)$$



Discrete Geometry

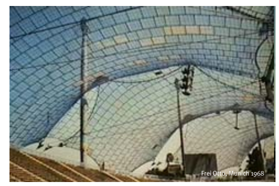
Basic tool

- differential geometry
  - metric, curvature, etc.



Discrete realizations

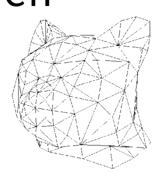
- “meshes”
  - computational geom.
  - graph theory



Discrete Diff.Geometry

Building from the ground up

- discrete geometry is the given
  - meshes: triangles, tets
  - more general: cell complex



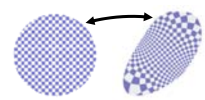
- how to do calculus?
  - preserve crucial properties

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Discr. Diff. Geometry

Building from the top down

- high level theorems
  - Riemann mapping
  - Willmore energy
  - Hamilton's prcple.



$f : \Omega_1 \subset \mathbb{C} \rightarrow \Omega_2 \subset \mathbb{C}$

$\boxed{\int_S (\kappa_1 - \kappa_2)^2 \, dA}$

$$0 = \delta_q \int_0^T L(q(t), \dot{q}(t)) dt$$

Is It Much Better? 

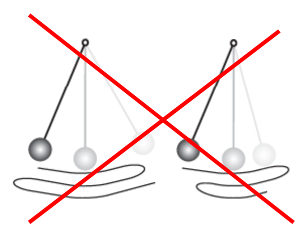
Magic happens

- much more robust

Is there a recipe?

- yes, but several ones…
- discrete var. principle

Discrete from the start



Discrete

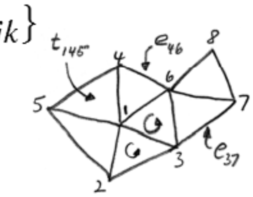
Setup

“pointers” “floats”

- topology & geometry
- simplicial complex: “triangle mesh”
  - 2-manifold $K = \{V, E, T\}$

    $V = \{v_i\}$ $E = \{e_{ij}\}$ $T = \{t_{ijk}\}$
  - Euler characteristic

    $F - E + V = 2(1-g) = \chi$



Mean Curvature Flow

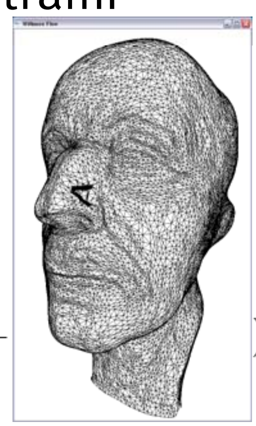
Laplace and Laplace-Beltrami

- Dirichlet energy

$$\begin{aligned}\Delta u &= 0\\ u|_{\partial\Omega} &= u_0\end{aligned} \rightsquigarrow \min \int (\nabla u)^2$$

- on surface

$$\partial_t v_i = -(H\vec{n})_i = -\frac{1}{4A_i}\sum_{e_{ij}}(\cot\alpha_{ij} + \quad)$$



Parameterization

Harmonic function

- from surface to region in $\mathbb{R}^2$

$$u : S \rightarrow \mathbb{R}^2$$

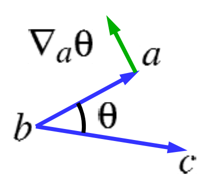
$$\Delta_S u = 0$$

- linear system
  - boundaries
    - Dirichlet/Neumann

Geometry-based Approach

Benefits

- everything is geometric
  - often more straightforward
  - tons of indices: forbidden!



The story is not finished...

- still many open questions
  - in particular: numerical analysis

What Matters?

Structure preservation!

- symmetry groups
  - rigid bodies: Euclidean group
  - fluids: diffeomorphism group
  - conformal geometry: Möbius group
- many more: symplectic, invariants, Stokes' theorem, de Rham complex, etc. (pick your favorite...)

Accuracy
Speed
Size

Themes for This Class

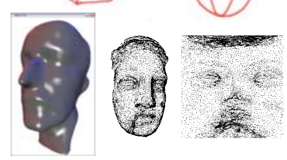
What characterizes structure(s)?

- dealing with shape
  - Euclidean invariance
- dealing with physics
  - conservation/balance laws
- importance of measures
  - mass, area, curvature, flux, circulation

The Program

Things we will cover

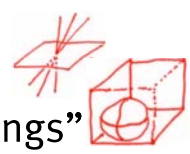
- warmup: curves
  - discrete analogues of cont. theorems
- surfaces: some basic operators
  - the discrete setting
  - putting them to work
    - denoising/smoothing, parameterization

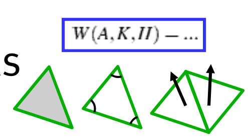


The Program

Things we will cover

- what can we measure
  - invariant measures of “things”
    - curvature integrals without derivatives
- a first physics model
  - deformation of a shape
  - simulating discrete shells

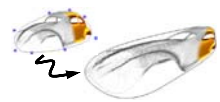




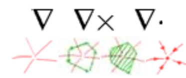
The Program

Things we will cover

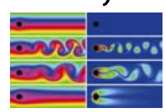
- interpolation on simplicial complexes, i.e., meshes



- discrete exterior calculus

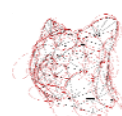


- putting it to work: discrete fluids
  - structure preservation: vorticity
    - ensured by design!



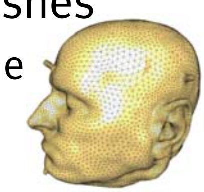
The Program

Things we will cover



- conformal geometry
  - conformal parameterizations
  - curvature energies

- how to make all those meshes
  - sampling a surface/volume
  - variational tet meshing



The Plan

Learn Discrete Differential Geometry

- what's involved
  - lectures
  - slides and lecture notes online
  - four or five hmw assignments
  - OR a research project
    - something that makes sense for you
    - should be decided within next 4 weeks

Maybe some external lecturers too (visitors)

The Bigger Picture

Maintain fundamental structures!

- symmetries
- invariants

Lots of stuff coming together

- Computational topology, algebraic topology, DEC, computational science, etc...