

THIN SHELLS & CURVATURE-BASED ENERGY

THIN SHELLS AND THIN PLATES

Thin, flexible objects
Shells are naturally *curved*
Plates are naturally *flat*

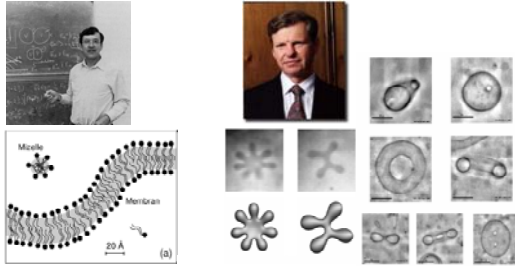


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PHYSICS OF MEMBRANES

S. Helfrich (FU Berlin), P. Canham (U.W. Ontario)

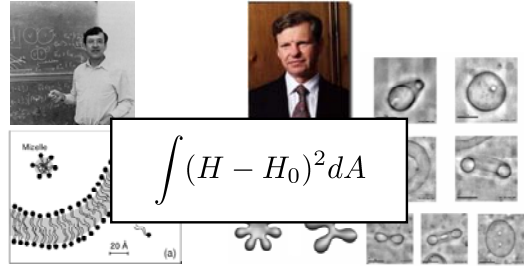


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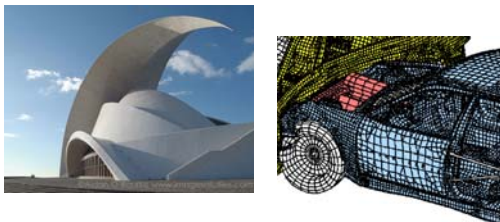


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ENGINEERING

Civil/mechanical/aeronautical design



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MATHEMATICS

T. J. Willmore's surfaces

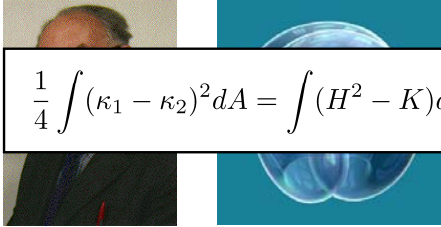


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MATHEMATICS

T. J. Willmore's surfaces



$$\frac{1}{4} \int (\kappa_1 - \kappa_2)^2 dA = \int (H^2 - K) dA$$

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MATHEMATICS

T. J. Willmore's surfaces



$$\frac{1}{4} \int (\kappa_1 - \kappa_2)^2 dA = \int (H^2 - \textcircled{K}) dA$$

$$\int H^2 dA = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA$$

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RELATED WORK

Researchers in graphics:

- Terzopoulos, Bridson, Breen, etc.
 - ad-hoc models for cloth
- Bobenko & Suris, Pai
 - discrete models of elastic curves



[Choi and Ko]

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EULER'S ELASTICA

Early formulation of elastic curves



$$E^{\text{bend}} = \int_0^l \kappa(s)^2 ds$$

Bernoulli began generalization to surfaces

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CHLADNI'S VIBRATING PLATES



Plate vibrated by violin bow

Sand settles on nodal curves

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CHLADNI'S VIBRATING PLATES

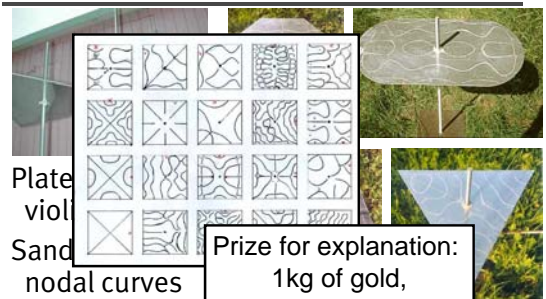


Plate vibrated by violin bow
Sand settles on nodal curves

Prize for explanation:
1kg of gold,
1808,1811,1815

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PROBLEM SETUP

What is the deformation energy?

undeformed body \bar{x} → deformation → deformed body x

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PROBLEM SETUP

undeformed config. is curved ∴ thin shell

What is the deformation energy?

undeformed body \bar{x} → deformation → deformed body x

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PROBLEM SETUP

undeformed config. is flat ∴ thin plate

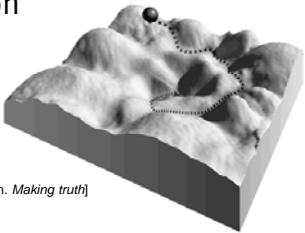
What is the deformation energy?

undeformed body \bar{x} → deformation → deformed body x

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PROBLEM SETUP

Energy is a non-negative scalar function



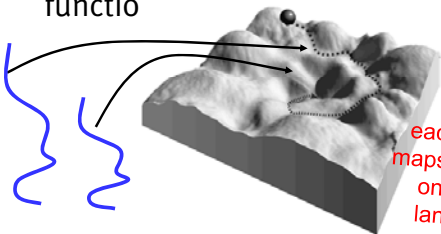
[T. L. Brown. *Making truth*]

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PROBLEM SETUP

Energy is a non-negative scalar function

real number, coordinate-frame invariant




each config. maps to a point on energy landscape

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PROBLEM SETUP

Internal forces push "downhill"



$f = -\nabla E$

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PLATES



Germain



Poisson



Navier

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THIN PLATE ENERGY

Germain's argument:

- bending energy must be a symmetric even fct of principal curvatures

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA$$

$$= \int H^2 dA$$

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THIN PLATE ENERGY

Poisson's linearization

- assuming small displacements, approx. curvature by 2nd derivatives

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA$$

$$E_{lin}^{bend} = \int (\Delta f)^2 dA$$

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THIN PLATE ENERGY

Navier's equation

- to find minimizer for linearized energy, solve a PDE

$$\Delta^2 f = 0$$

$$E_{lin}^{bend} = \int (\Delta f)^2 dA$$

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THIN PLATE ENERGY

Navier's equation

- to find minimizer for linearized energy, solve a PDE

$$\Delta^2 f = 0$$

$$E_{lin}^{bend} = \int (\Delta f)^2 dA$$

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AXIOMATIC APPROACH

Energy should be:

- symm., even fct of princ. curvatures
- extrinsic measure
- smooth w.r.t. change in shape
- invariant under rigid-body motion
- simple to compute
- easy to understand

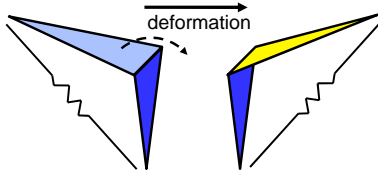
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WHAT ABOUT MASS/SRING?

Diagonal springs *don't* work

- reference configuration is *curved*
- incorrect energy minima



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AXIOMATIC SHELLS

“Simplest” answer to desiderata

$$(H - H_0)^2$$

Derivation:

extrinsic change in *shape* operator

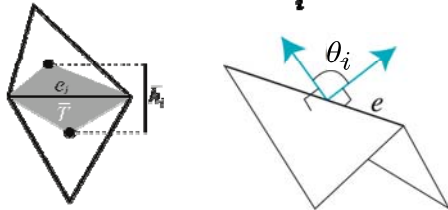
$$[\text{Tr}(\varphi^*S) - \text{Tr}(\bar{S})]^2$$

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DISCRETE SHELLS

$$\text{Elastic energy} = \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\|\bar{e}_i\|}{\bar{h}_i}$$



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DISCRETE SHELLS

$$\text{Elastic energy} = \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\|\bar{e}_i\|}{\bar{h}_i}$$

Gradient gives forces:

$$f_k = K_B \sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i} (\bar{\theta}_i - \theta_i) \nabla_{x_k} \theta_i$$

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PIMP A CLOTH SIMULATOR

Have a cloth simulator handy?

- reuse all the existing code
- retrofit the bending term
- precompute ref. quantities offline

$$f_k = K_B \sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i} (\bar{\theta}_i - \theta_i) \nabla_{x_k} \theta_i$$

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MODELING PAPER

Paper sheet

- curled
- creased
- pinned



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ARE WE DONE?

Discrete shells are nice and simple.

What else is out there?



Kirchhoff



Love



Karman



Koiter



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