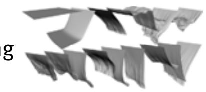
Simulation with Meshes

Animation

Example: Cloth

- time evolution of a mesh subject to
  - internal forces
    - stretch and bending
  - external forces
    - gravity, drag, user applied forces
- setting it up



From Volino & Magnenat-Thalmann

Setup

Vertices as functions of time

- position, velocity, acceleration

$$x_i \quad \dot{x}_i \quad \ddot{x}_i = M^{-1} f(x, \dot{x})$$

- time dependence (first order)

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ M^{-1} f(x,v) \end{pmatrix}$$

Discretization

Time stepping

- forward Euler $\begin{pmatrix} \Delta x \\ \Delta v \end{pmatrix} = dt \begin{pmatrix} v_0 \\ M^{-1} f_0 \end{pmatrix}$
- backward Euler

$$\begin{pmatrix} \Delta x \\ \Delta v \end{pmatrix} = dt \begin{pmatrix} v_0 + \Delta v \\ M^{-1} f(x_0 + \Delta x, v_0 + \Delta v) \end{pmatrix}$$

  - implicit equation! BUT: stable!

Implicit Solution

Simple approach

- linearize f (Taylor series to 1st order)

$$f(x_0 + \Delta x, v_0 + \Delta v) = f_0 + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial v}\Delta v$$

$$\begin{pmatrix} \Delta x \\ \Delta v \end{pmatrix} = dt \begin{pmatrix} v_0 + \Delta v \\ M^{-1}(f_0 + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial v}\Delta v) \end{pmatrix}$$

$$\Delta v = dt M^{-1}(f_0 + \frac{\partial f}{\partial x}(dt(v_0 + \Delta v)) + \frac{\partial f}{\partial v}\Delta v)$$

Time Stepping

Final equation

$$\left(I - dt M^{-1}\frac{\partial f}{\partial v} - dt^2 M^{-1}\frac{\partial f}{\partial x}\right)\Delta v = dt M^{-1}\left(f_0 + dt\frac{\partial f}{\partial x} v_0\right)$$

- now just need f...
- classic approach: potential energy
  - force follows as negative gradient

Energies and Forces

Approaches

- continuum models
  - discretization through finite elements/volumes/differences
- discrete models
  - simplest example: mass/spring systems

Baraff & Witkin

Constraints as key element

- forces given directly
  - in a moment...
- equilibrium as vanishing condition

$$C(x) = 0$$

- ancillary energy

numerical device (not physics!)

$$\frac{k}{2} C(x)^T C(x) \longrightarrow f_i = -k \frac{\partial C(x)}{\partial x_i} C(x)$$

constraint forces

Forces Continued

Need further derivatives

- stiffness matrix

$$K = \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial v} = 0$$

$$K_{ij} = \frac{\partial f_i}{\partial x_j} = -k \left(\frac{\partial C(x)}{\partial x_i} \frac{\partial C(x)}{\partial x_j}^T + \frac{\partial^2 C(x)}{\partial x_i \partial x_j} C(x) \right)$$

C(x) Functions

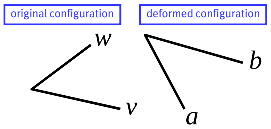
What do we want?

- resist stretch and shear
- measure with the deformation tensor

original configuration

deformed configuration

$$f(v) = a \quad f(w) = b$$

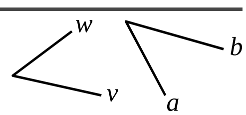


$$C(x) = \left(\partial f \partial f^T \right) - I$$

$$\phi : T_e \rightarrow T_g \quad \psi : T_e \rightarrow T_h$$

Deformation Gradient

$$f = \psi \circ \phi^{-1}$$
$$\partial f = \partial \psi \partial \phi^{-1}$$



$$\partial \psi^T \partial \psi = \begin{pmatrix} \langle a,a \rangle & \langle a,b \rangle \\ \langle b,a \rangle & \langle b,b \rangle \end{pmatrix}$$

$$\phi : T_e \rightarrow T_g \quad \psi : T_e \rightarrow T_h$$

$$\partial \phi^{-1} \partial \phi^{-1T} = \begin{pmatrix} \langle w,w \rangle & -\langle v,w \rangle \\ -\langle v,w \rangle & \langle v,v \rangle \end{pmatrix} \frac{1}{(\det \partial \phi)^2}$$

constant...

Damping Forces

Necessary for simulation

- in direction of force
- proportional to velocity

gradient in direction of velocity

$$\dot{C}(x) = \frac{\partial C(x)}{\partial x}^T \dot{x}$$

$$d = -k_d \frac{\partial C(x)}{\partial x} \dot{C}(x)$$

$$f = -k_s \frac{\partial C(x)}{\partial x} C(x)$$

compare to

direction

magnitude

- Hessian

Damping Forces

Hessian

- get rid of non-symmetric term

$$\frac{\partial d_i}{\partial x_j} = -k_d \left(\cancel{\frac{\partial C(x)}{\partial x_i} \frac{\partial \dot{C}(x)}{\partial x_j}^T} + \frac{\partial^2 C(x)}{\partial x_i \partial x_j} \dot{C}(x) \right)$$

- depends on velocity

$$\frac{\partial d_i}{\partial v_j} = -k_d \frac{\partial C(x)}{\partial x_i} \frac{\partial \dot{C}(x)}{\partial v_j}^T = -k_d \frac{\partial C(x)}{\partial x_i} \frac{\partial C(x)}{\partial x_j}^T$$

Oy Ve…

What else?

- actual constraints…
  - point constraints easy
  - (let's just leave it at that for now)

Bending

- much smaller component but can be important

Bending

Guess what: dihedral angle

- next time