# Parameterization of Meshes



## 2-Manifold

What makes for a smooth manifold?

- locally looks like Euclidian space
- collection of charts
  - mutually compatible on their overlaps
- form an atlas

Parameterizations are key

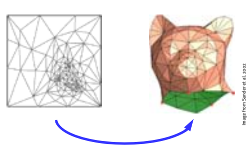


## Parameterizations

What is a parameterization?

- function from some region $\Omega \subset \mathbb{R}^2$ to the embedded surface $M \subset \mathbb{R}^3$

$$S(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

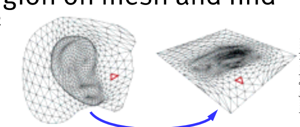




## Task

Find map from mesh to $\mathbb{R}^2$

- identify region on mesh and find map to $\mathbb{R}^2$



- what's different?
  - existence of solutions when mapping to convex region



## Parameterizations

Applications

- texture mapping
  - piecewise linear interpolation of attributes across mesh

- resampling

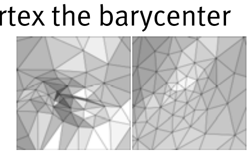
- simulation
  - needs surface as function



## First Attempt

Tutte, 1963

- make planar vertex the barycenter of its neighbors
- fix boundary
  - convex!




$$\forall i \notin \partial G : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_i = \frac{1}{N_i} \sum_{\{j | e_{ij} \in E\}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_j$$

## Conventions

How do we denote things?

- we are looking for 2 scalar valued functions defined over the surface

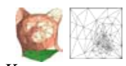
  $u(x) = (u_1(x_1,x_2,x_3), u_2(x_1,x_2,x_3))$

- we will quietly ignore the dimensionality of $u$ and $x$ vectors
  - though it will be obvious from context



## Computing it

Solution of linear system



- input: $x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+K}$
- boundary: convex K-gon $u_{n+1},\ldots,u_{n+K}$
- edge weights: $\lambda_{ij} = \begin{cases} \lambda_{ij} \geq 0 & e_{ij} \in E \\ 0 & e_{ij} \notin E \end{cases}$

$u_i = \sum_{j=1}^{n+K} \lambda_{ij} u_j$

$\forall i: \sum_{j=1}^{n+K} \lambda_{ij} = 1$

$i = 1,\ldots,n$

$(I - \Lambda)u = b$



## Existence

Solvable?

- must show: $(I-\Lambda)w = 0 \Rightarrow w = 0$
  - i.e., no redundancy in equations
- what solver?
  - system is large, (non-)symmetric
  - iterative
    - Jacobi, Gauss-Seidel
    - CG, bi-CG



## Setting the Weights

How to choose "barycenter"

- Tutte, "uniform"
  - minimizes square lengths (springs)

$$F(u_1,\ldots,u_n) = \sum_{\{j|e_{ij}\in E\}} w_{ij}||u_i - u_j||^2$$

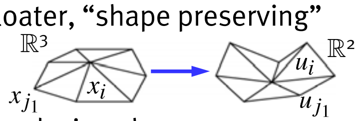
$$\frac{\partial F}{\partial u_i} = 2 \sum_{\{j|e_{ij}\in E\}} w_{ij}(u_i - u_j) \qquad \lambda_{ij} = \frac{w_{ij}}{\sum_{\{j|e_{ij}\in E\}} w_{ij}}$$



## Setting the Weights

How to choose the "barycenter"

- Floater, "shape preserving"




- geodesic polar map

Lengths: $||x_{j_k} - x_i|| = ||u_{j_k} - u_i||$

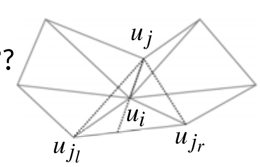
Scaled angles: $\angle(u_{j_k}, u_i, u_{j_{k+1}}) = \frac{\angle(x_{j_k}, x_i, x_{j_{k+1}})}{\sum_{\{j|e_{ij}\in E\}} \angle(x_j, x_i, x_{j+1})}$



## Setting the Weights

Floater weights

- barycentric coords??

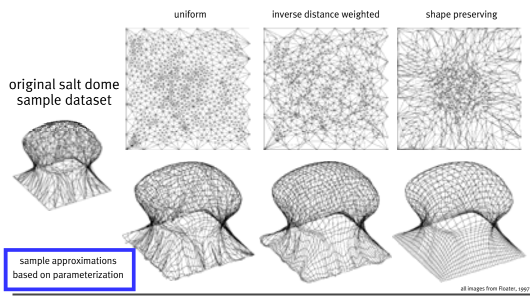



$$\delta_j = \frac{\text{area } u_i, u_{j_l}, u_{j_r}}{\text{area } u_j, u_{j_l}, u_{j_r}}$$

$$u_i = \delta_j u_j + \delta_{j_l} u_{j_l} + \delta_{j_r} u_{j_r}$$

$$\forall j: u_i = \sum_l \mu_{lj} u_l \quad \rightarrow \quad \lambda_{ij_l} = \frac{1}{N_i} \sum_j \mu_{lj}$$

## How well does it Work?

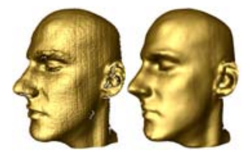





## Smoothing Connection

Denoising surfaces

- move towards centroid in $\mathbb{R}^3$
- connection with Laplacian smoothing
  - Taubin, 1995
  - Desbrun, 1999
  - Guskov, 1999





## Laplace Equation

Why is this connection useful?

- measures smoothness
  - second derivatives (gradient squared)
  - over "time", minimizes it

Question

- how to discretize Laplace

$$\nabla^2 x \approx Lx$$



## Laplace Discretization

Taubin, 1995

- uniformity assumption

$$Lx_i = \frac{1}{N_i} \sum_{\{j|e_{ij}\in E\}} x_j - x_i$$

Umbrella Operator

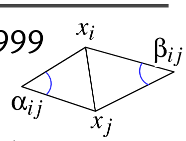
- smoothing of both
  - geometry
  - parameterization



## Laplace Discretization

With collaborators, in 1999

- Laplace-Beltrami
  - mean curvature flow
  - take shape into account



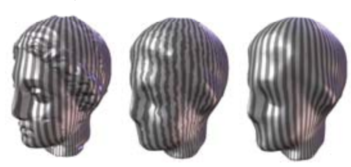

$$\begin{aligned} -H_i\vec{n}_i &= \nabla_{x_i} A \\ &= \frac{1}{4A_i} \sum_{\{j|e_{ij}\in E\}} (\cot\alpha_{ij} + \cot\beta_{ij})(x_j - x_i) \end{aligned}$$



## Mean Curvature Flow

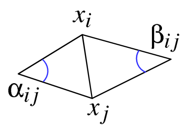
With collaborators, in 1999

- also used implicit time stepping for stability



# Measuring Distortion

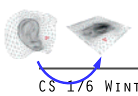
Energy of a map
- discrete harmonic
  - Eck, Polthier, Desbrun




domain

$$E_{A_i}(u) = \sum_{\{j|e_{ij}\in E\}} (\cot\alpha_{ij} + \cot\beta_{ij})||u_j - u_i||^2$$

range



$$\int_T |\nabla s|^2 du = \int_T \left(\frac{\partial s}{\partial u_1}\right)^2 + \left(\frac{\partial s}{\partial u_2}\right)^2 du$$



# Harmonic Map

Properties
- area minimization leads to minimal surfaces: soap bubbles

$$\left\langle \frac{\partial s}{\partial u_1}, \frac{\partial s}{\partial u_2} \right\rangle = 0$$

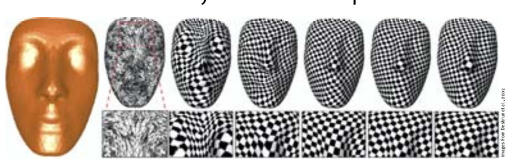
$$\left|\frac{\partial s}{\partial u_1}\right| = \left|\frac{\partial s}{\partial u_2}\right|$$

$$\begin{aligned} A &= \frac{1}{2}\int_T \left|\frac{\partial s}{\partial u_1} \times \frac{\partial s}{\partial u_2}\right| du \\ &\leq \frac{1}{2}\int_T \left|\frac{\partial s}{\partial u_1}\right| \left|\frac{\partial s}{\partial u_2}\right| du \\ &\leq \frac{1}{4}\int_T \left(\frac{\partial s}{\partial u_1}\right)^2 + \left(\frac{\partial s}{\partial u_2}\right)^2 du \end{aligned}$$



# Good Measures

Comparison of methods
- from Desbrun et al., 2002
  - sensitivity to mesh shape






# Variational Approach

Find parameterization which minimizes discrete energies
- DHP (harmonic)

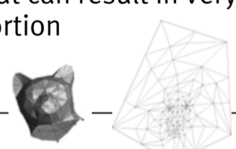
$$\frac{\partial E_A}{\partial u_i} = 0 = \sum_{\{j|e_{ij}\in E\}} (\cot\alpha_{ij} + \cot\beta_{ij})(u_j - u_i)$$

  - symmetric linear system... (nice)
  - need to fix boundary still



# DHP

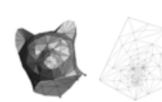
Properties
- used to build discrete conformal map
- note that coefficients can be negative (bad!)
- keeps angles but can result in very large area distortion





# Boundaries

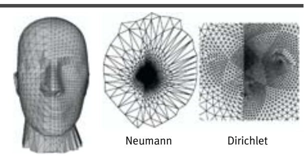
How to fix them?
- Dirichlet boundary
  - interpolation
  - e.g., k-gon on circle with relative boundary edge length preserved
  - “unnatural”
- Neumann boundary
  - match derivatives on boundary



## Natural Boundary

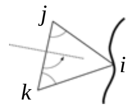
Area gradient

- match on boundary




$$\sum_{\{j,k|(i,j,k)\in T\}} \cot\alpha_k(u_i-u_j)+\cot\alpha_j(u_i-u_k)$$

$$= \sum_{\{j,k|(i,j,k)\in T\}} R^{\pi/2}(u_k-u_j)$$






## Computational Issues

Computing charts

- face clustering
  - all manner of issues...
- boundary conditions
  - map to particular boundary? which??

Linear system solvers

- iterative for large, sparse systems!



## Recent Results

More formal def. of conformality

- see Schröder et al.

Non linearity creeps in

- but much stronger properties!