Fluid Mechanics

Fluid Models (I)

Euler Equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = -\nabla p + f$$

$$\nabla \cdot u = 0$$

- inviscid fluids (not viscous)
- incompressible
- non-linear PDE, with linear constraint

Fluid Models (II)

Navier-Stokes Equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u = -\nabla p + f - \nu\, \Delta u$$

$$\nabla \cdot u = 0$$

- only change: viscosity
  - coefficient ν
- loss of total energy during motion

Algorithm for Simulation

One of many possibilities... (see CFD lit.)
"Stable Fluids" (Stam 99)

- adapted for graphics needs
- regular Eulerian discretization

$u_{ij}^{(t)} \in \mathbb{R}^3$

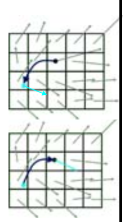
$$u_{ij}^{*} = u_{ij}^{(t)} + \Delta t\, f_{ij}^{(t)} - [\Delta t (u \cdot \nabla) u]_{ij}^{(t)}$$

$$\Delta q_{ij} = \nabla \cdot u_{ij}^{*} \qquad u_{ij}^{(t+\Delta t)} = u_{ij}^{*} - \nabla q_{ij}$$

Implementation Issues

Advection $[\Delta t (u \cdot \nabla) u]_{ij}^{(t)}$

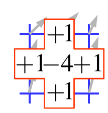
- discretize? Nah...
  - non-linear and nasty
- method of characteristics
  - parcels transported along velocity...
  - let's go backwards in time
    - to know where a "parcel" is *coming from*
    - need to interpolate velocities
    - and resample them
  - unconditional stability!
    - large time step; but artificial viscosity....



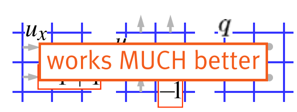
What Where?

Co-located grids

- velocities & pressures at vertices



- staggered grids



"Geometry" of Fluids? $\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p$

Euler equations seem clear
- advection + div-free projection ad infinitum
  - Stam's Stable Fluids do this wonderfully well
  - numerous follow-up work (Fedkiw *et al.*)
- but what does it mean, geometrically?
  - "total energy" is rather unintuitive
  - is there a notion of momentum preservation?

Yes
- but of course, we need to massage the PDE
- so as to reveal the geometric structure

Geometry Revealed $\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p$

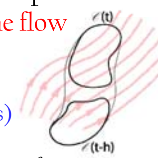
Pressure disappears when we take the curl:

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \quad \text{(vorticity)}$$

$$\nabla \cdot u = 0 \qquad u \parallel \partial \mathcal{D}$$

- vorticity measures the "spin" of a parcel
- vorticity is "advected" along the flow
- the circulation around *any* *closed loop* is **constant** as it gets advected (by Stokes)
  - known as *Kelvin's theorem*
  - call it preserv. of angular momentum if you want

$\Gamma(t) = \oint_{\mathcal{C}(t)} u.dl$



Geometry Revealed

So we know:

Integral of vorticity constant on advected sheet

Additionally, **ω** *defines* ***u***
- if we ignore complex topology for a moment
- $u = \nabla \times (\Delta^{-1}\omega)$ because u is divergence free!

Vorticity is the only real variable here and Kelvin's is ***a defining property***

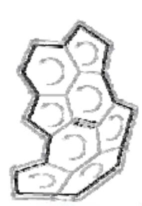
(Navier-Stokes: loss along the way)

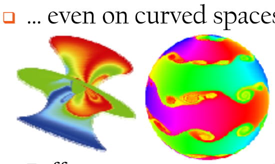
Towards a Proper Discretization

Domain discretization = *simplicial complex*
- fluxes through faces for velocity
  - intrinsic (coordinate-free) and **eulerian**
    - » reminiscent of staggered grids…
- net flux for divergence
  - what comes in…must come out
- flux spin for vorticity
  - Torque created on a "paddle wheel"
- valid for any grid…

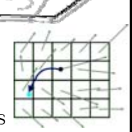
Discrete Kelvin's Theorem

Guarantees circulation preservation… for any discrete loop!
- big loop = union of small ones
- … even on curved spaces





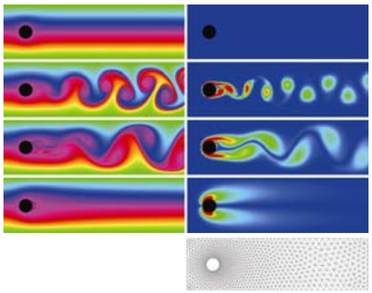
- Difference with Stable Fluids?
  - trace back integrals, not point values



Results

New method
- exact discrete vorticity preservation
- arbitrary simplicial meshes
  - see also [Feldman *et al.* '05, Bargteil *et al* '06]
- **everything is intrinsic**
- basic operators very simple (super parse)
- great flows for small meshes!
  - computationally efficient even on coarse mesh
  - no need for millions of vortex particles

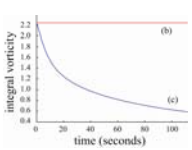
Channel

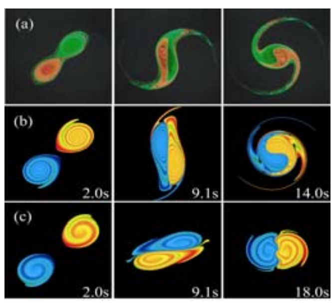


Smoking Bunny

7k vertices, 32k tets; 0.45s per frame on PIV (3GHz)

Merging Vortices





Movie