Introduction to
Artificial Intelligence

Lecture 8 – Logical reasoning

CS/CNS/EE 154

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Logics in general

- Logics are formal languages for representing information such that conclusions can be drawn
- Syntax defines the sentences in the language
- Semantics defines the "meaning" of sentences, i.e., the truth of a sentence in a world (environment state)

- *Example*: Language of arithmetic

$3 + * 4 ==$ not well-formed

$3 + 4 = 6$ well formed, but false

$3 + x = 6$ true in world $\{(x, 3)\}$
false in world $\{(x, 2)\}$
true in world $\{(x,3), (y,2)\}$

$3 = 3$ true in all worlds

Models

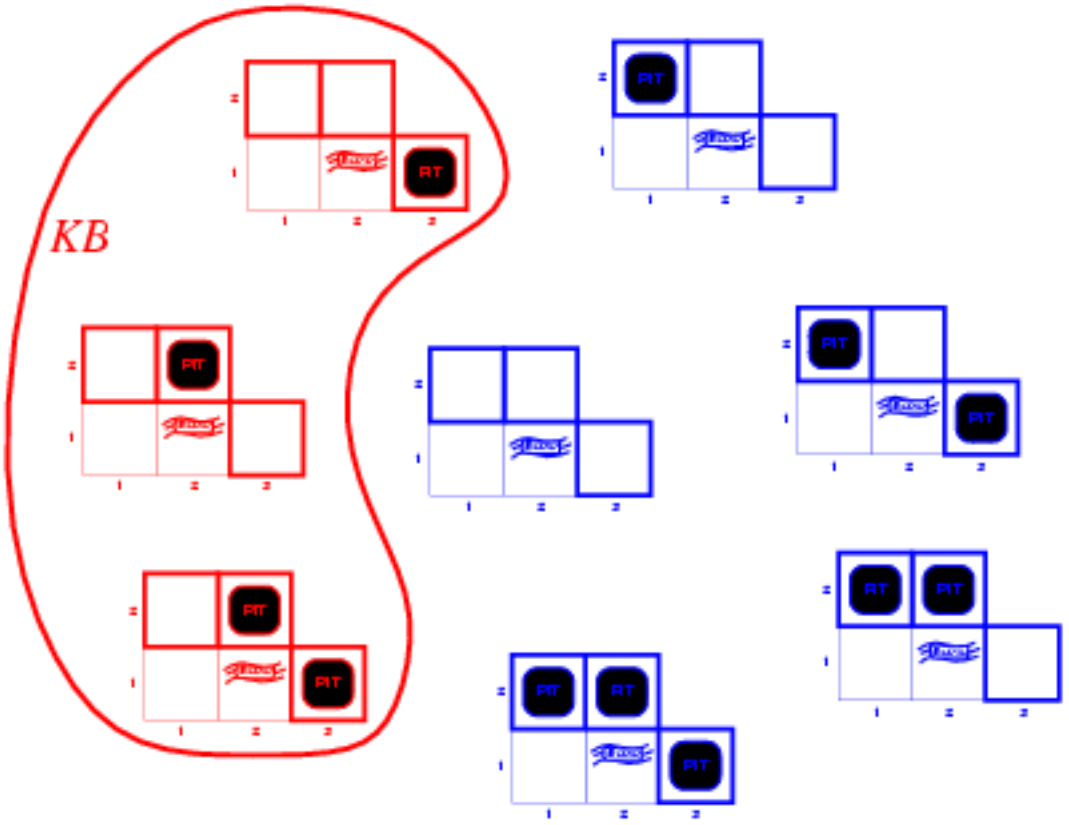
- Logicians think in terms of models
  - Formally structured worlds w.r.t. which truth can be evaluated
- We say *m* is a model of a sentence *α* if *α* is true in *m*

  $\alpha \equiv (x + 2 = 5)$ is true in model $m = \{(x, 3)\}$
- M(α) is the set of all models of α

  $\alpha \equiv (x + 2 = y)$ $M(\alpha) = \{ \{(x, 0), (y, 2)\}, \{(x, 3), (y, 5)\}, \ldots \}$
- Then $KB \models \alpha$ if and only if

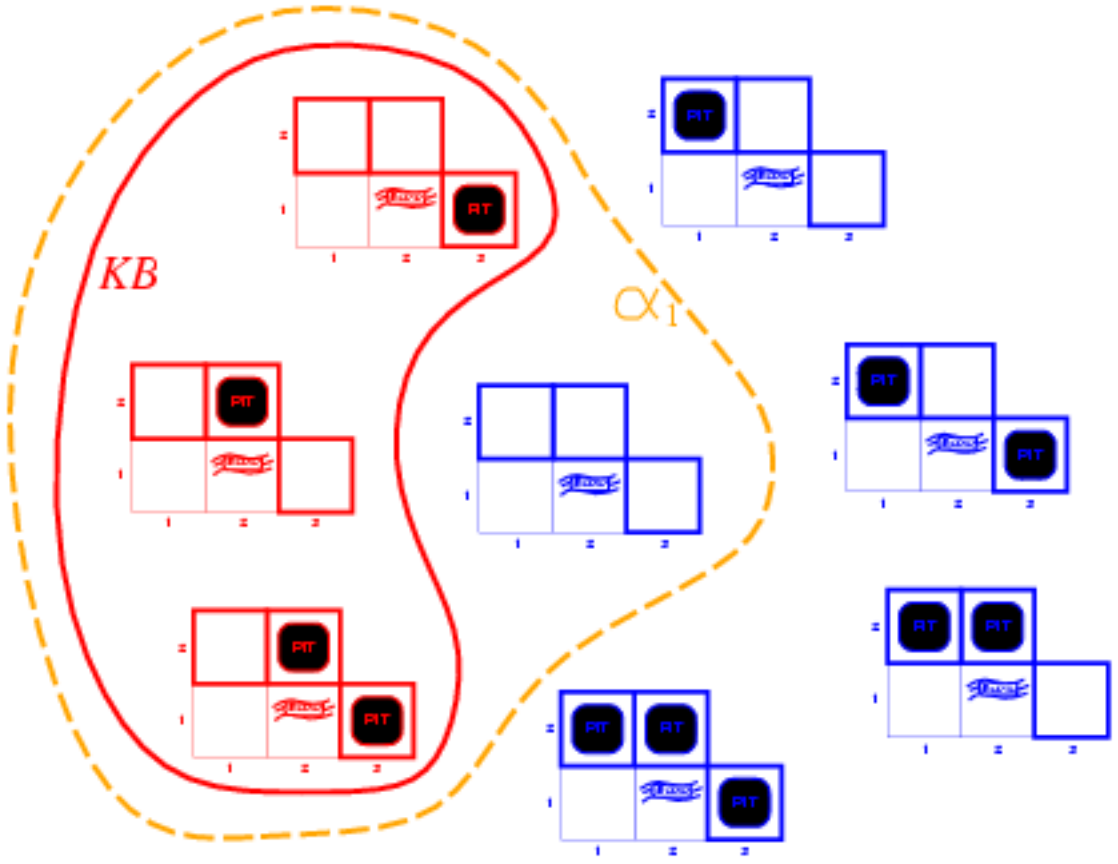
$$M(KB) \subseteq M(\alpha)$$

Wumpus models



- *KB* = wumpus-world rules + observations

Wumpus models



- *KB* = wumpus-world rules + observations
- α_1 = "[1,2] is safe", $KB \models \alpha_1$

Propositional logic: Syntax

- Simplest example of a logic; illustrates basic ideas

- Propositional symbols are sentences
- If S is a sentence, $\neg S$ is a sentence (negation)
- If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)
- If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)
- Notation shorthand:
  - $S_1 \Rightarrow S_2$ for $\neg S_1 \vee S_2$ (implication)
  - $S_1 \Leftrightarrow S_2$ for $(S_1 \Rightarrow S_2) \wedge (S_2 \Rightarrow S_1)$ (biconditional)

Propositional logic: Semantics

Each model specifies *true* or *false* for each proposition symbol

E.g.

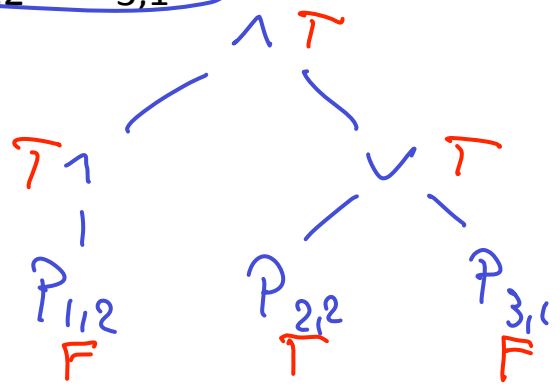
$P_{1,2}$	$P_{2,2}$	$P_{3,1}$
false	true	false

Rules for evaluating truth with respect to a model *m*:

$\neg S$	is true iff	S is false		
$S_1 \wedge S_2$	is true iff	S_1 is true	and	S_2 is true
$S_1 \vee S_2$	is true iff	S_1 is true	or	S_2 is true

Simple recursive process evaluates an arbitrary sentence, e.g.,

$\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1})$



Wumpus world in prop. logic

Let $P_{i,j}$ be true if there is a pit in [i, j].

Let $B_{i,j}$ be true if there is a breeze in [i, j].

$KB = \{ \neg P_{1,1},\ \neg B_{1,1},\ B_{2,1} \}$ $\quad KB \equiv \neg P_{1,1} \wedge \neg B_{1,1} \wedge B_{2,1}$

"Pits cause breezes in adjacent squares"

$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$
$$B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$$

Proving entailment

- Two main classes of methods for proving $KB \models \alpha$

- Model checking
  - Truth table enumeration (always exponential in n)
  - Better: CSP (e.g, improved backtracking such as DPLL)
    Check whether ($KB \wedge \neg \alpha$) is unsatisfiable

- Proof using inference
  - Apply sequence of inference rules (syntactic manipulations)
  - Can use inference rules in a standard search algorithm

Logical inference

- Inference: procedure *i* for deducing (proving) sentences from knowledge base
- We say $KB \vdash_i \alpha$ if α can be inferred from KB using inference procedure *i*
- Inference i is called
  - Sound if whenever $KB \vdash_i \alpha$ then also $KB \models \alpha$
  - Complete if whenever $KB \models \alpha$ then also $KB \vdash_i \alpha$

- Thus, a sound and complete inference procedure *correctly* answers *any* question whose answer can be inferred from *KB*

Resolution

- Assumes sentences in Conjunctive Normal Form (CNF)
  - This is no restriction (Tseitin transformation)
  - Example $(P_{11} \vee \neg B_{12}) \wedge (B_{12} \vee P_{11} \vee P_{22}) \wedge \ldots$

- Resolution inference rule

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \quad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \ell_k \vee m_1 \vee m_{j-1} \vee m_{j+1} \cdots \vee m_n}$$

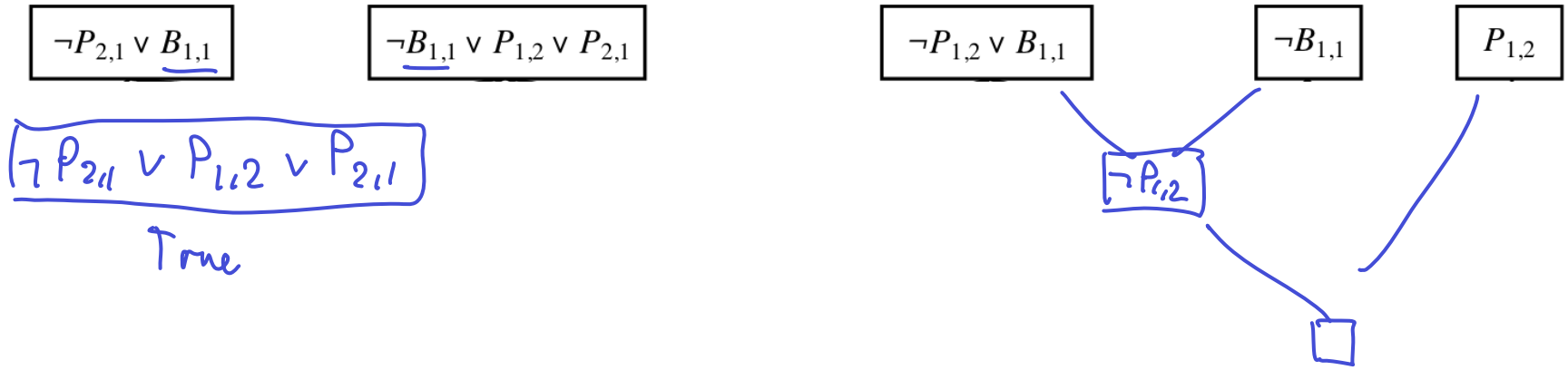
- Sound and complete for propositional logic!
- Example:

$$\frac{\text{Rainy} \Rightarrow \text{WetLawn} \quad , \quad \text{WetLawn} \Rightarrow \text{Slippery}}{\neg R \vee W \qquad\qquad \neg W \vee S}$$

$$\neg R \vee S \equiv \text{Rainy} \Rightarrow \text{Slippery}$$

Resolution example

- $KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$ $\quad \alpha = \neg P_{1,2}$



Thus $KB \models \alpha$

□

Logical reasoning with resolution

- Resolution is complete

  ➔ Any propositional sentence is entailed if and only it can be proven by resolution

- BUT: Finding the proof can be difficult!
  - Must search through possible applications of resolution rule
  - Search space exponentially large

- 3CNF SAT is NP complete!
  - Existence of polynomial time algorithm considered unlikely

- Are there special kinds of sentences that are “easy” to prove??

Horn clauses

- Special types of propositional formulae
- A Horn clause is
  - A propositional symbol; or
  - (conjunction of symbols) ⇒ symbol

Grades ∧ GRE ∧ Statement ∧ Letter ⇒ Grad School
Research ⇒ Letter
SURF ⇒ Research
} Horn

Study ⇒ GRE ∧ Grades } NOT Horn

≡ ¬ Study ∨ (GRE ∧ Grades)

≡ (¬ Study ∨ GRE) ∧ (¬ Study ∨ Grades) } 2 Horn Clause

(¬ Study ∨ GRE): Study ⇒ GRE

Forward and backward chaining

- Inference procedure for special types of KBs, consisting only of Horn clauses

- Modus ponens complete for Horn formulas ☺

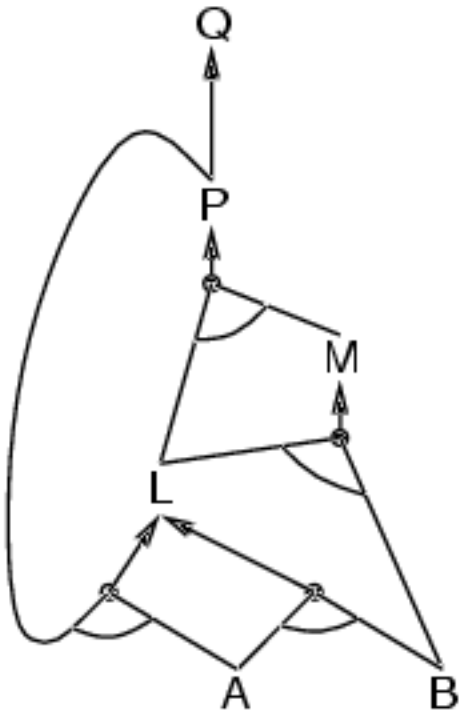
$$\frac{\alpha_1, \ldots, \alpha_k, \quad \alpha_1 \wedge \cdots \wedge \alpha_k \Rightarrow \beta}{\beta}$$

- Inference algorithms: forward and backward chaining

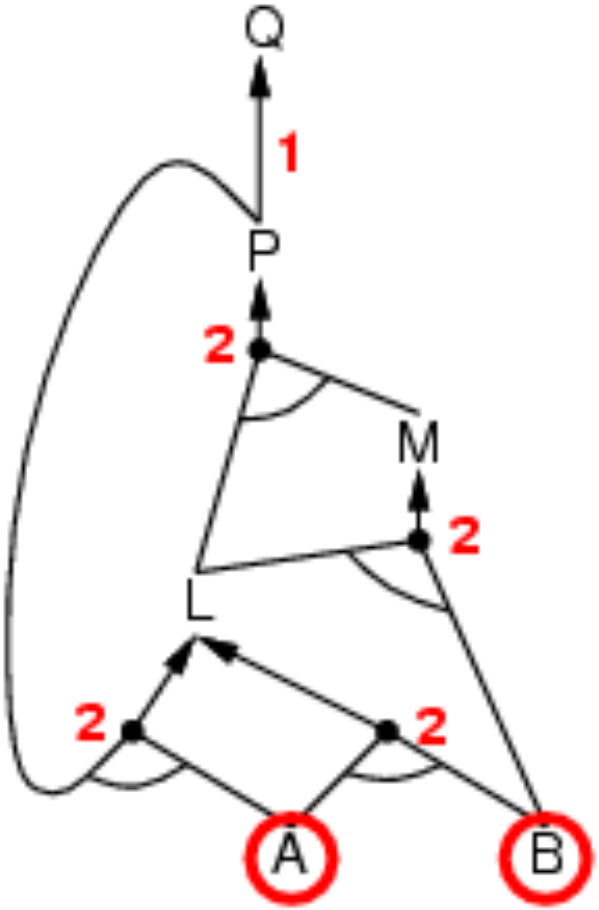
Forward chaining

- *Idea*: fire any rule whose premises are satisfied in the *KB*,
  - add its conclusion to the *KB*, until query is found

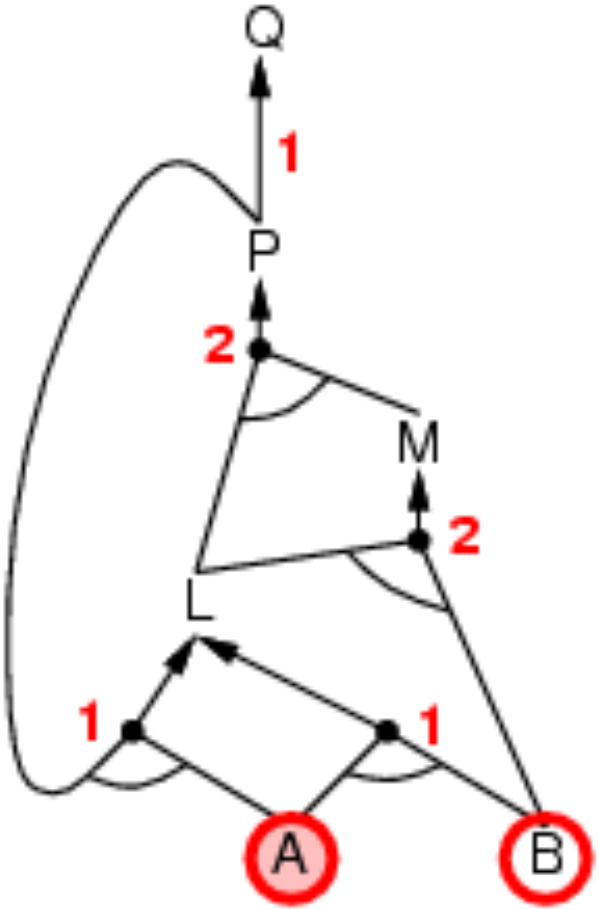
$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$



Forward chaining example



Forward chaining example



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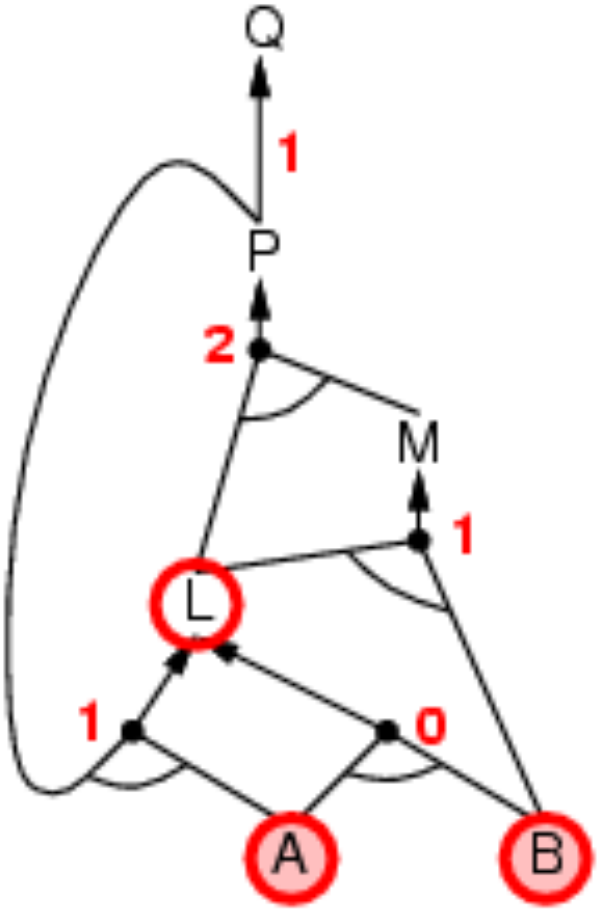
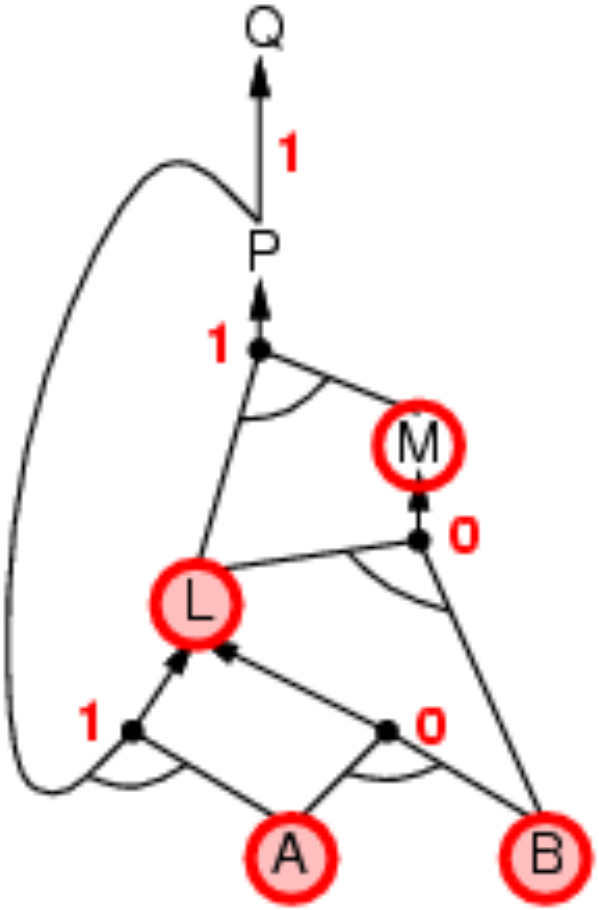
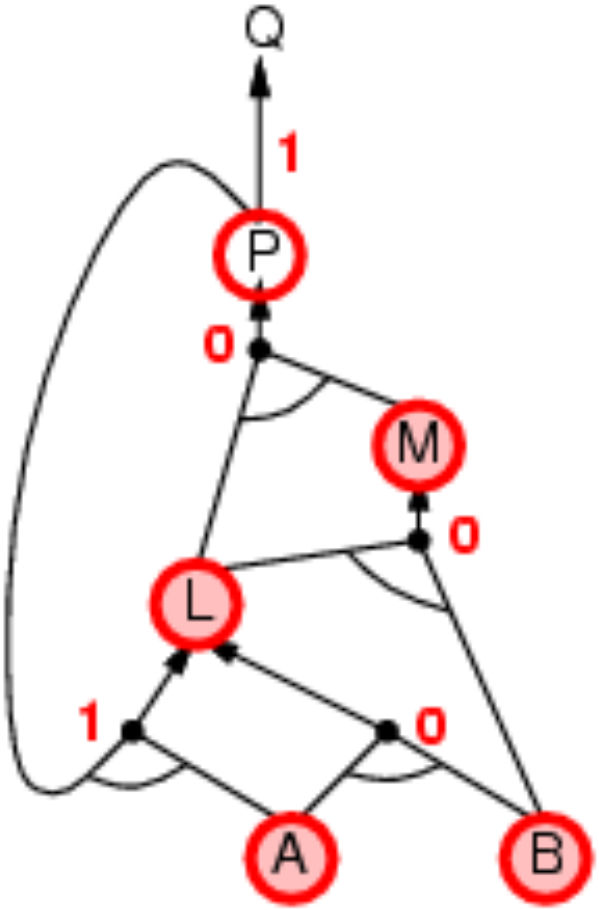


Forward chaining example



Forward chaining example



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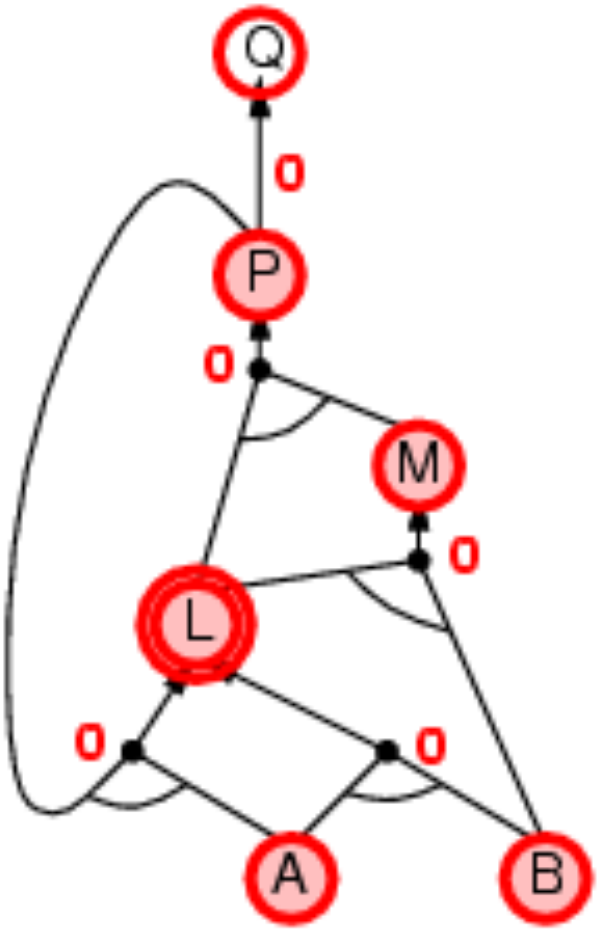

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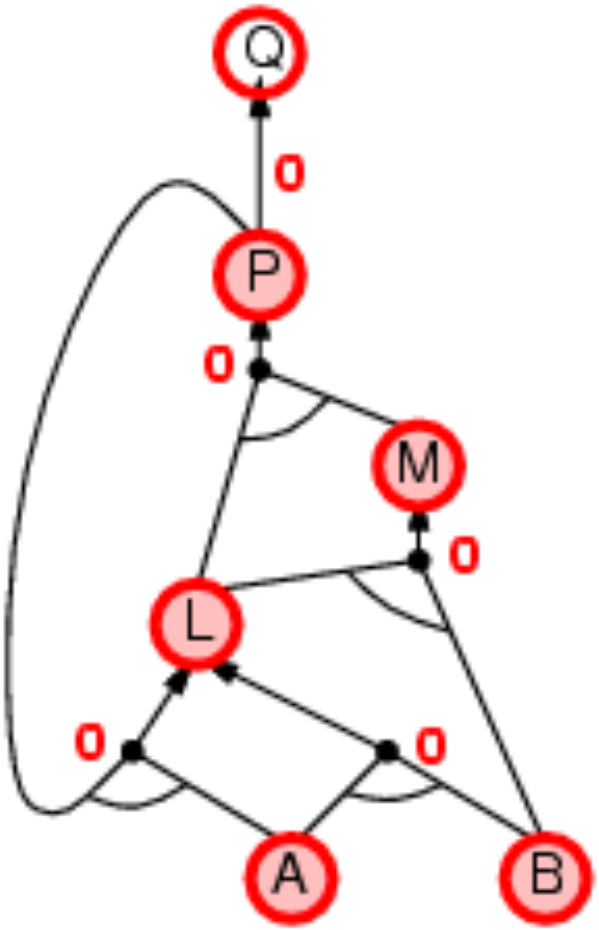

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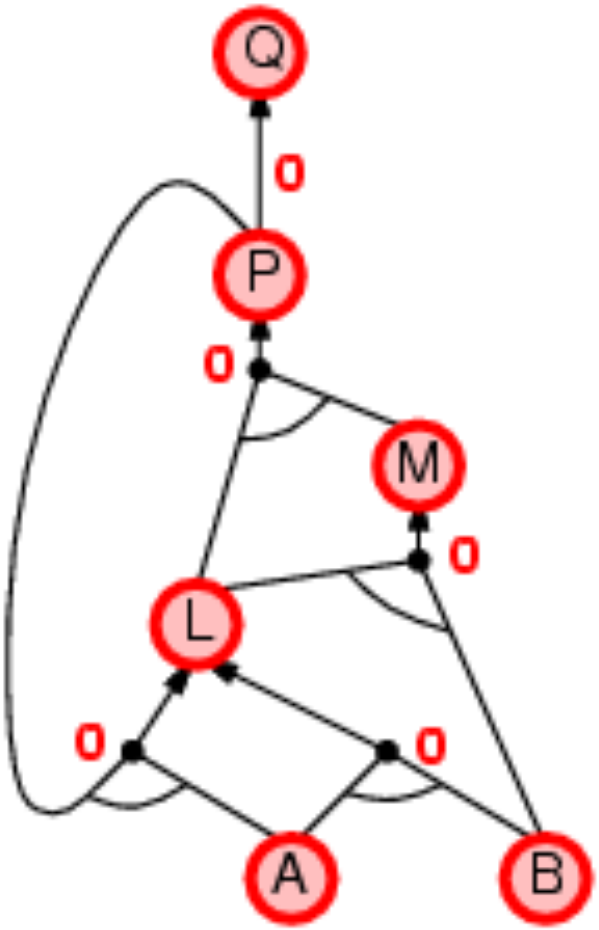


Proof of completeness

FC derives every atomic sentence that is entailed by *KB*

1. FC reaches a fixed point: no new atomic sentences are derived
2. Consider final state as model *m*, assigning true/false to symbols
3. Every clause in the original *KB* is true in *m*

   $a_1 \wedge \ \ldots \wedge \ a_k \Rightarrow b$
4. Hence *m* is a model of *KB*
5. If $KB \models q$, *q* is true in every model of *KB*, including *m*

Backward chaining

Idea: work backwards from the query *Q*:

check if *Q* is known already, or

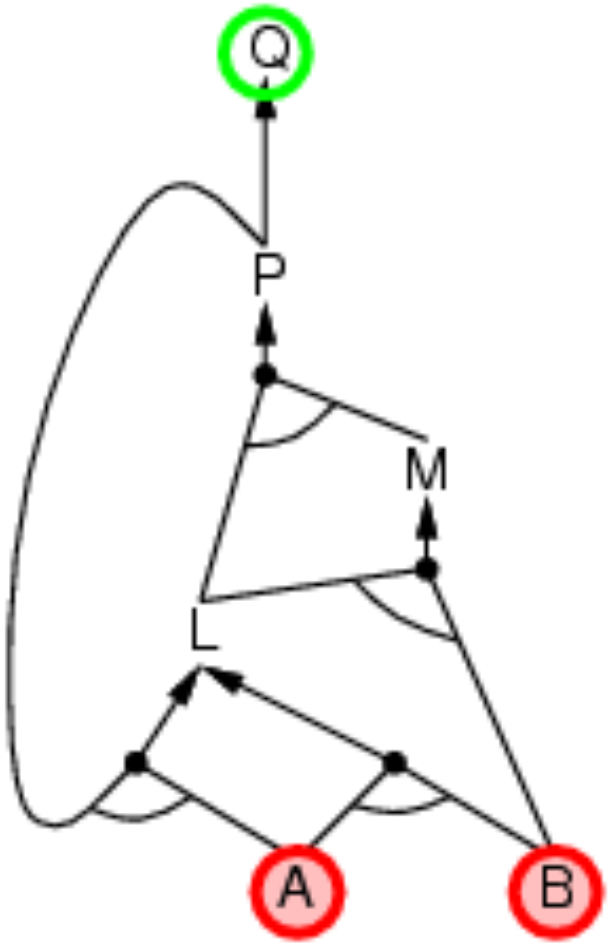
prove by BC all premises of some rule concluding *Q*

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1. has already been proved true, or
2. has already failed

Backward chaining example



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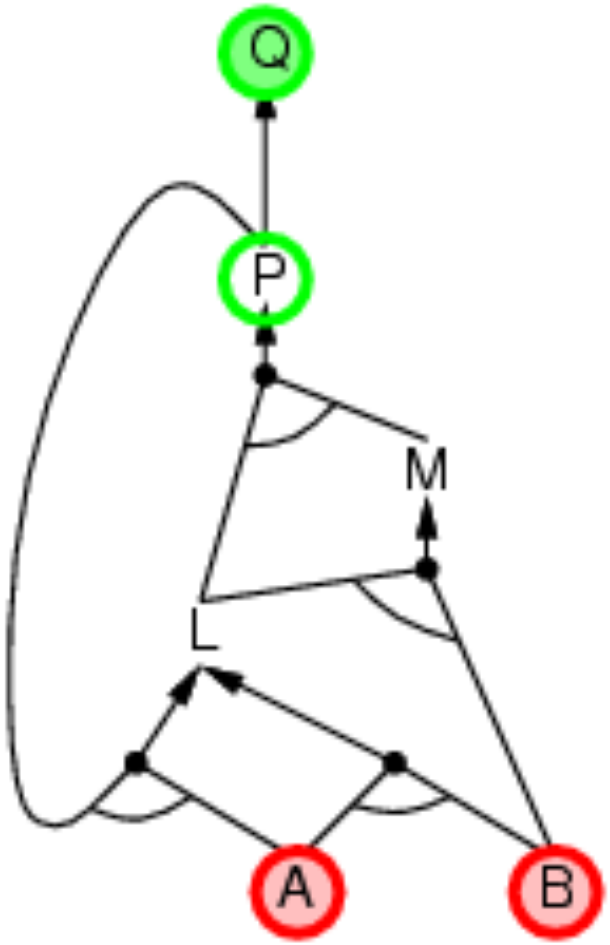

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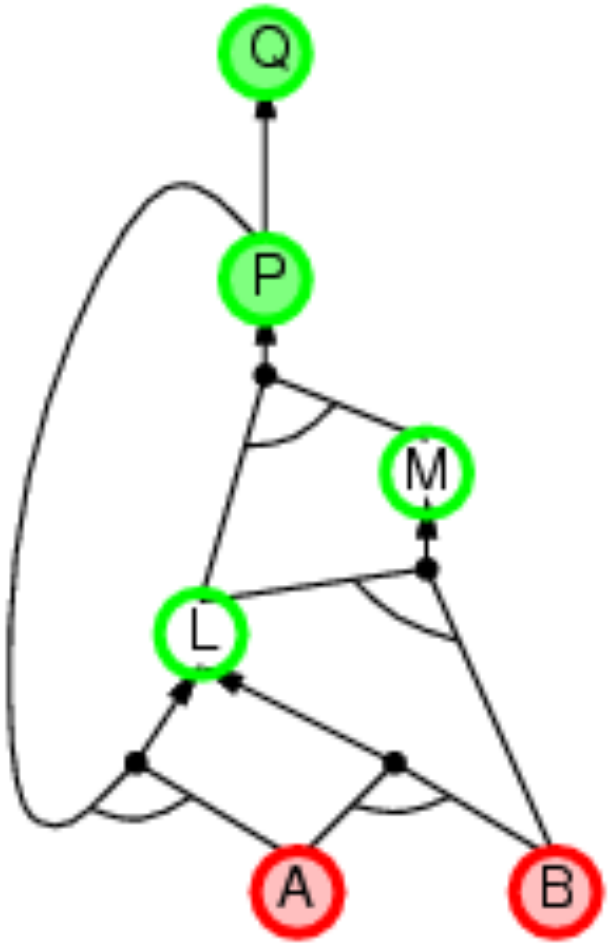

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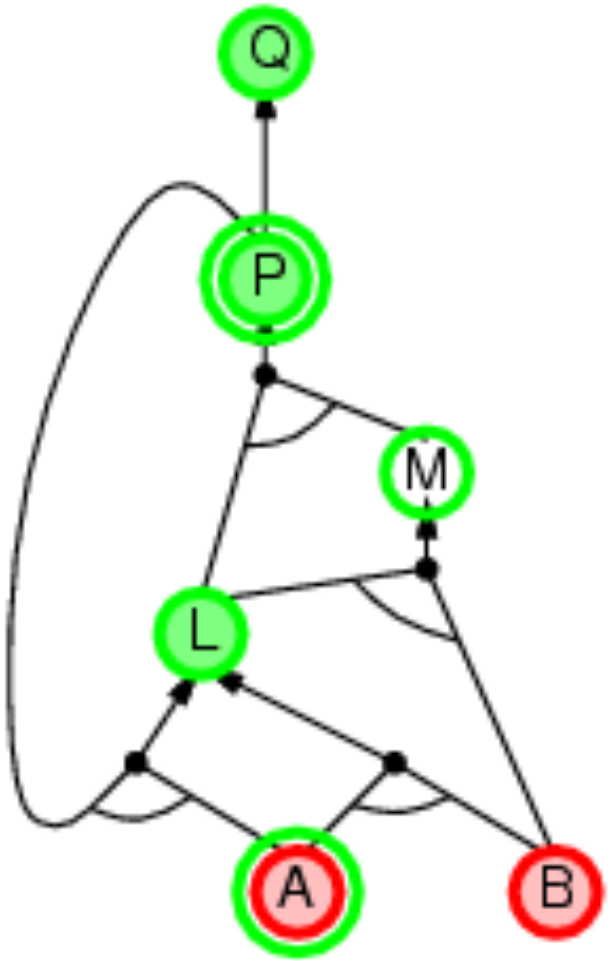

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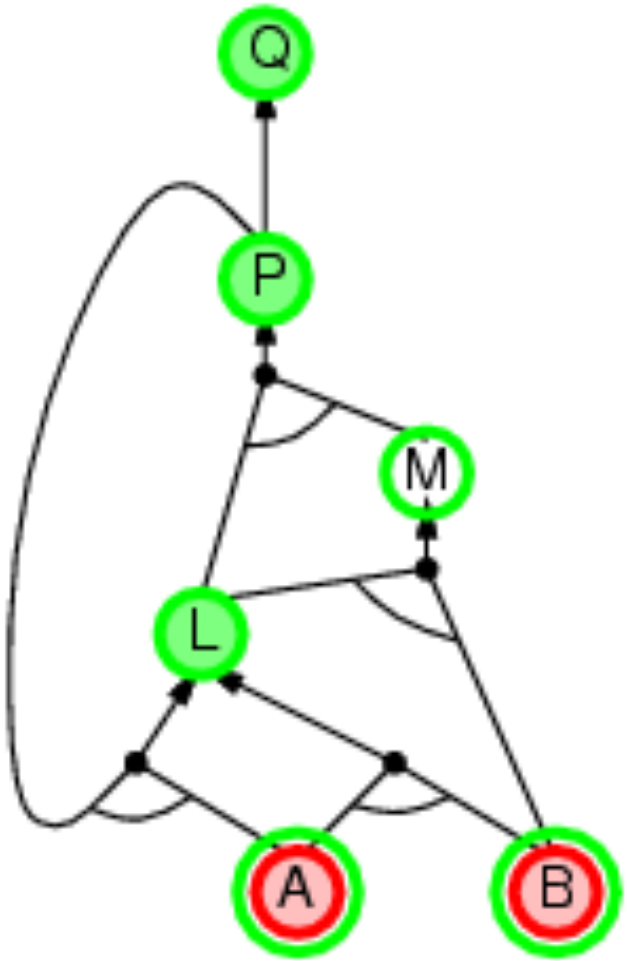

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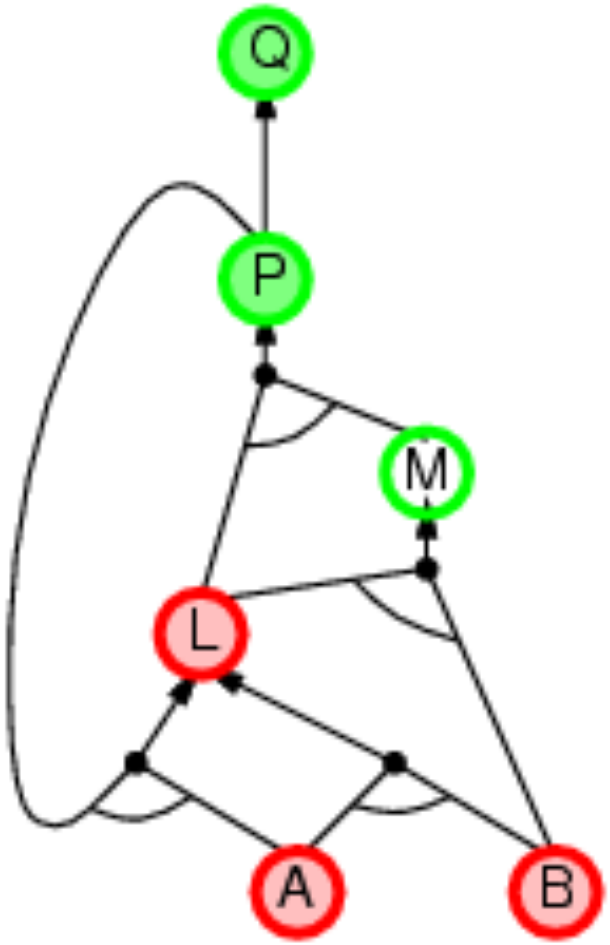

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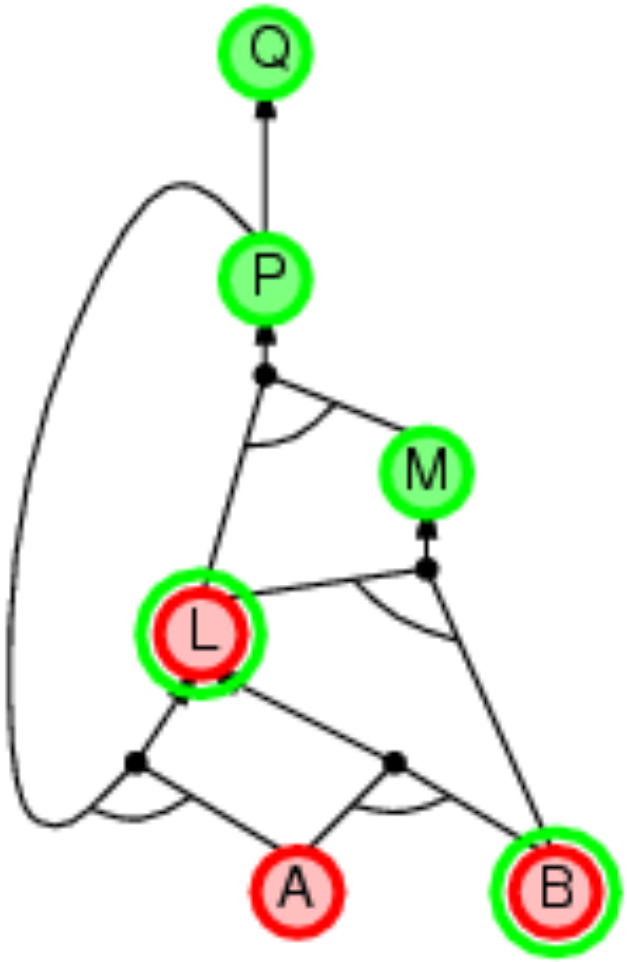

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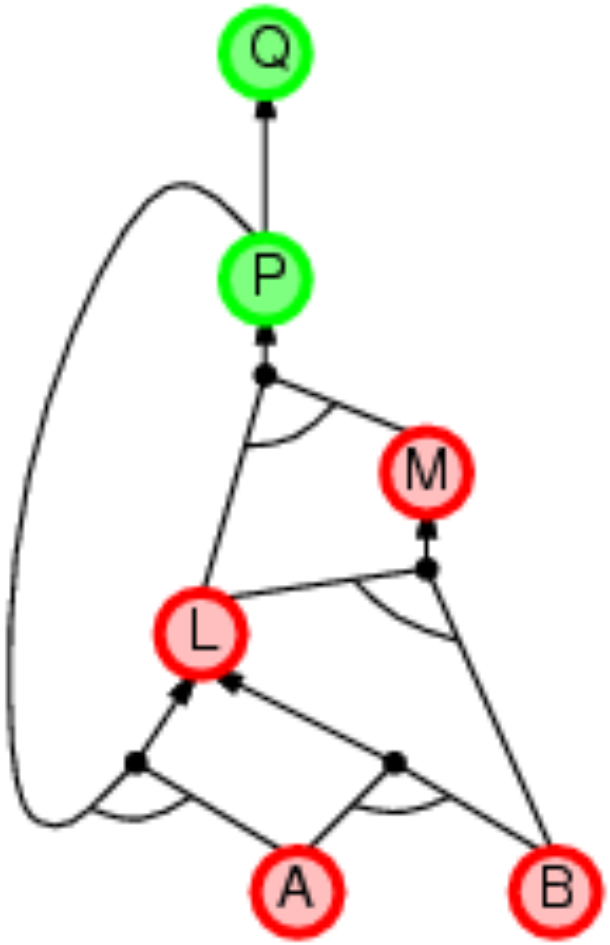

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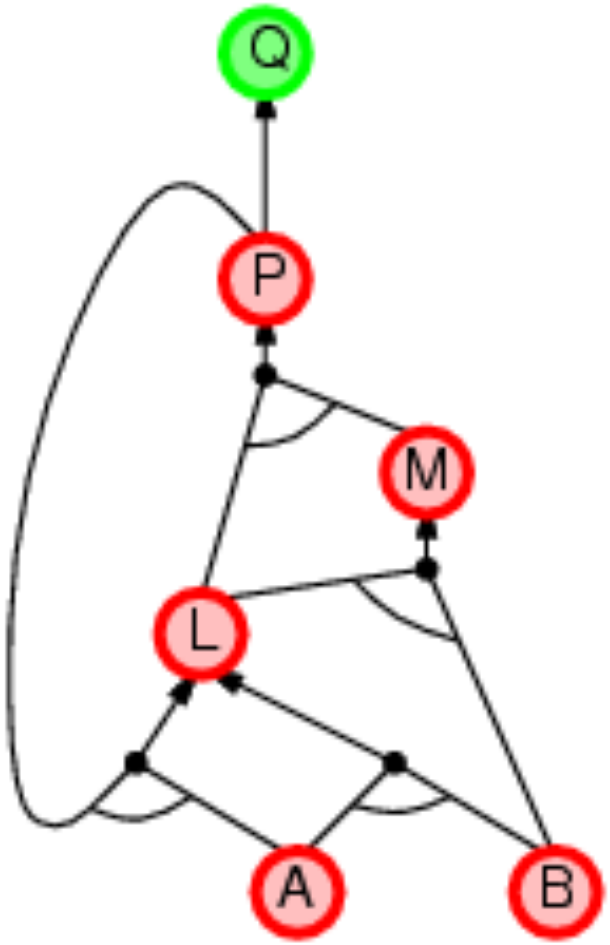

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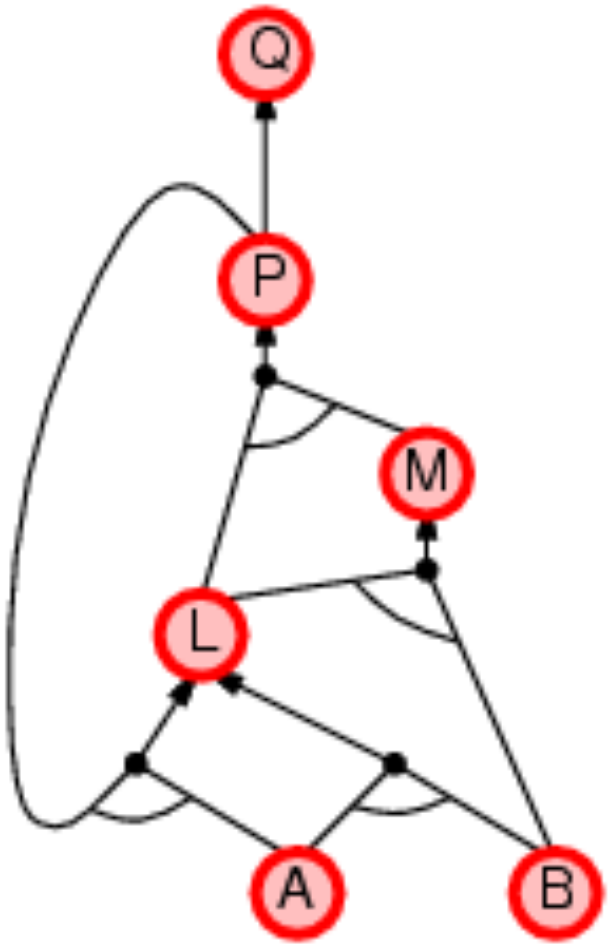


Forward vs. backward chaining

- FC is data-driven, automatic, unconscious processing,
  - e.g., simple model for object recognition, routine decisions
- May do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be much less than linear in size of KB

Summary so far:

- Logic = formal language with
  - Syntax (what sentences are valid?)
  - Semantics (what valid sentences are true?)
- Simple example: Propositional logic
- Can infer entailment of sentences using
  - Model checking (e.g., Constraint satisfaction)
  - Logical inference (should be sound and complete)
- Inference procedures
  - Resolution: Sound and complete for arbitrary prop. formulas, but exponential search space
  - Forward-/Backward chaining: Sound; complete only for Horn formulas. Inference in (sub-) linear time!

Issues with propositional Wumpus world

Need "copies" of symbols and sentences for each cell

$P_{1,1}$ is true if there is a pit in [1,1]
$P_{1,2}$ is true if there is a pit in [1,2]
…
$P_{n,n}$ is true if there is a pit in [n,n]

$B_{1,1}$ is true if there is a breeze in [1, 1] …
$B_{n,n}$ is true if there is a breeze in [n, n]
$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$; $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$; …

First order logic (FOL)

- Propositional logic is about simple facts
  - "There is a breeze at location [1,2]"
- First order logic is about facts involving
  - *Objects*: Numbers, people, locations, time instants, …
  - *Relations*: Alive, IsNextTo, Before, …
  - *Functions*: MotherOf, BestFriend, SquareRoot, OneMoreThan, …
- Will be able to say:
  - IsBreeze(x); IsPit(x); IsNextTo(x,y)

$$\forall x, y : (IsPit(x) \wedge IsNextTo(x, y)) \Rightarrow IsBreeze(y)$$

Simple example

- About King Richard the Lionheart and his evil brother John
- Objects:
  - Richard
  - John
  - Crown
- Relations
  - Richard and John are brothers
  - Richard is a king
- Function
  - Refer to specific properties of Richard and John, e.g., their head, legs, …

FOL: Basic syntactic elements

- Constants: KingJohn, 1, 2, …, [1,1], [1,2], …,[n,n], …
- Variables: x, y, z, …
- Predicates: Brother, >, =, …
- Functions: LeftLegOf, MotherOf, Sqrt, …
- Connectives: $\wedge$, $\vee$, $\neg$
- Quantifiers: $\forall$, $\exists$

- Constant, predicates and functions are mere symbols (i.e., have no meaning on their own)

FOL Syntax: Atomic sentences

A (variable-free) term is a

- constant symbol or
- k-ary function symbol: *function(term$_1$,term$_2$, …, term$_k$)*

Example: *LeftLegOf(KingJohn), ~~IsBreeze([1,2])~~*

Richard

Succ(Succ(… (Succ(Zero))))

An atomic sentence is a predicate symbol applied to terms

Example:

- *Brother(KingJohn, RichardLionheart)*
- *IsNextTo([1,1],[1,2])*
- *> (Length(LeftLegOf(KingJohn)), Length(LeftLegOf (RichardLionheart)))*

FOL Syntax: Composite sentences

- Composite sentences are
  - Atomic sentences or
  - Composite sentences joined by connectives

- Example:

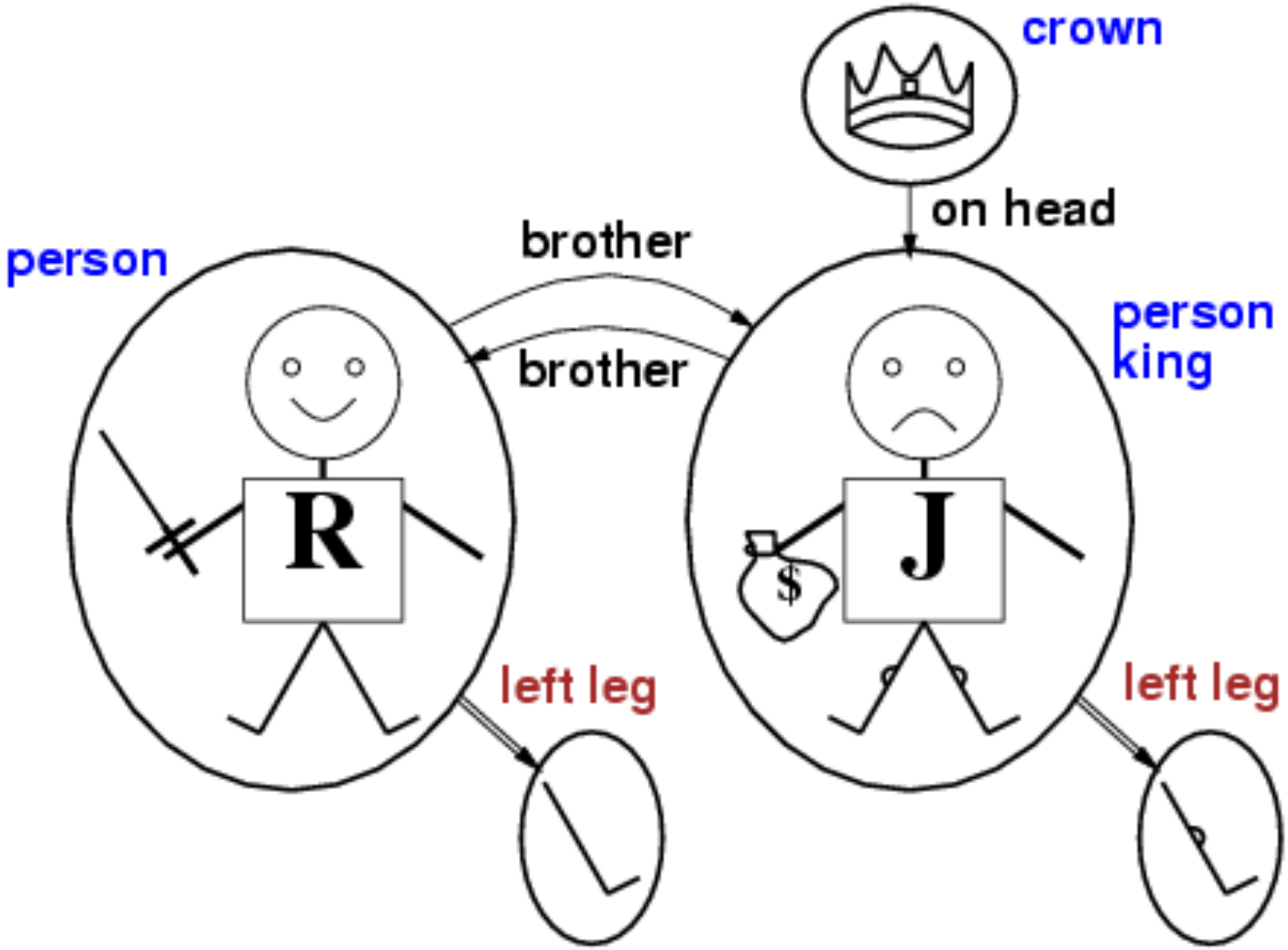
$$BrotherOf(KingJohn, RichardLionheart) \Rightarrow BrotherOf(RichardLionheart, KingJohn)$$

Models in FOL

- Much more complicated than in Propositional Logic
- Models contain
  - Set of objects (finite or countable)
  - Set of relations between objects (map obj's to truth values)
  - Set of functions (map objects to other objects)

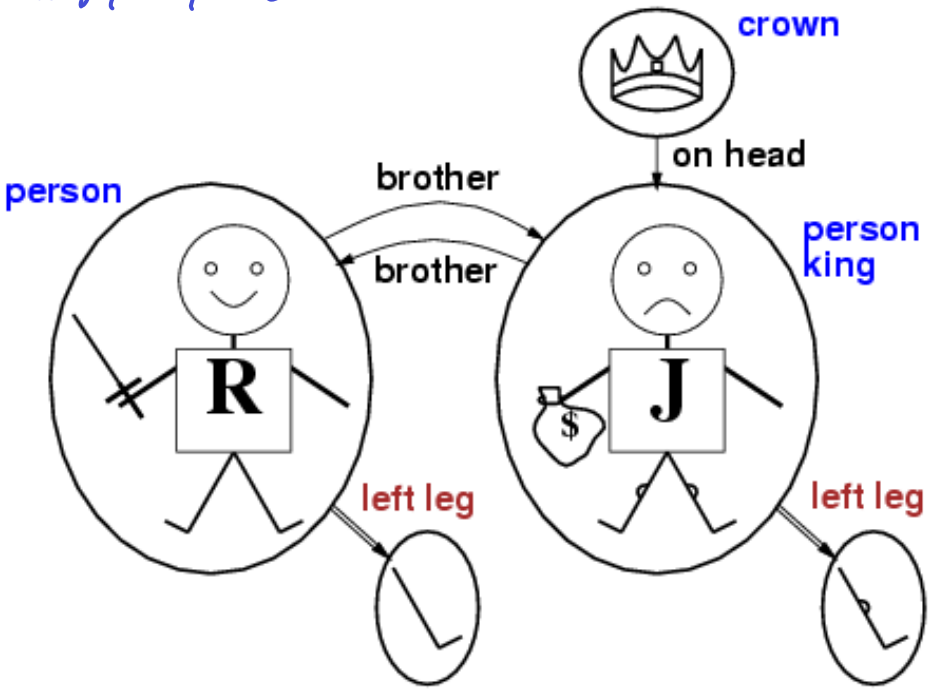
  and their interpretations:
  - Mapping from constant symbols to model objects
  - Mapping from predicate symbols to model relations
  - Mapping from function symbols to model functions
- An atomic sentence *predicate*($term_1$,$term_2$, …, $term_k$) is true if the objects referred to by $term_1$,$term_2$, …, $term_k$ are in the relation referred to by *predicate*

Models in FOL: Example



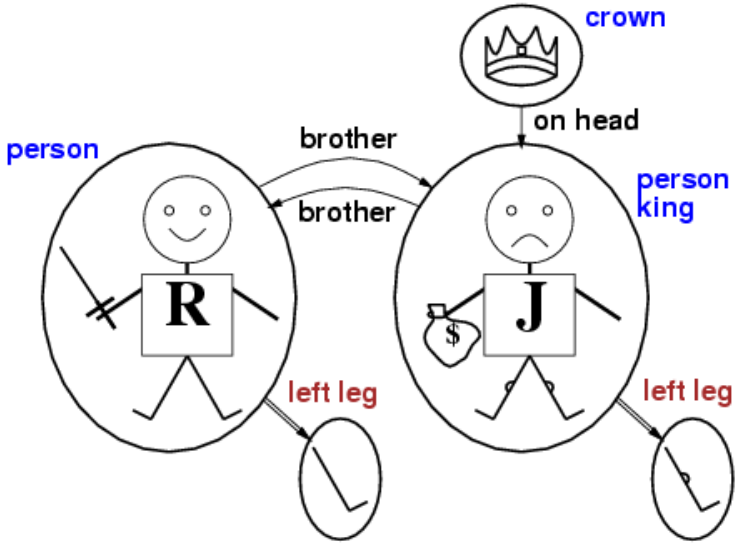
Models in FOL: Example

- *Objects*: R, J, C, LegR, LegJ, N
- *Functions*: LLO
  - LLO(R)=LegR; LLO(J)=LegJ; LLO(C) = N; LLO(LegR) = N; …
- *Relations*:
  - ~~B={R,J}~~; OH={(C,J)}; B = {(R,J), (J,R)}
    K={J}; P={R,J}
- *Mappings*:
  - Richard: R; John: J
  - LeftLegOf: LLO;
  - Brother: B; OnHead(OH)



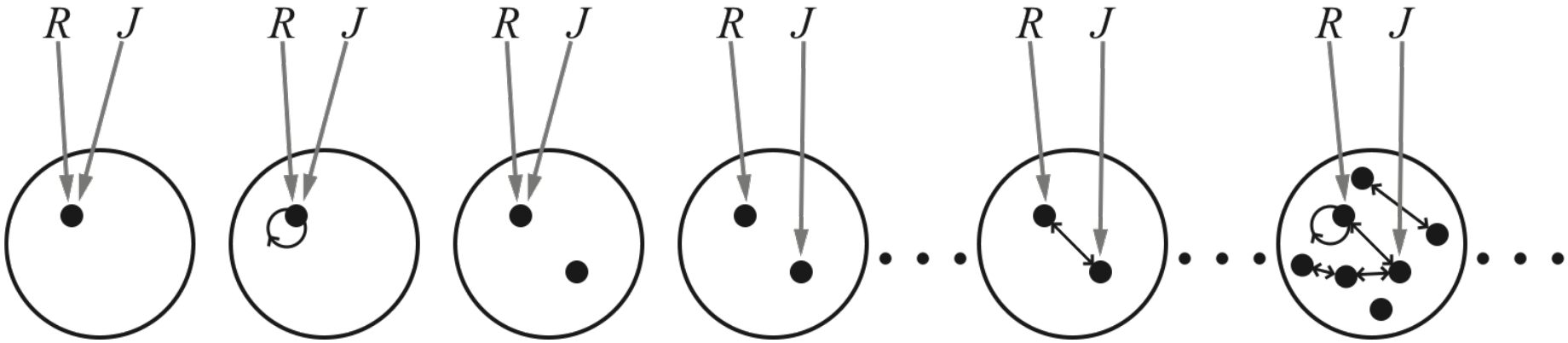
Subtleties with FOL models

- Specifying known facts is tedious
- E.g., need
  - ¬ OnHead(R,J)
  - ¬ OnHead(LeftLeg1,J)
  - ¬(R=J)
  - ¬ OnHead(LeftLeg1, LeftLeg2)
  - ¬(R=LeftLeg1)
  - …



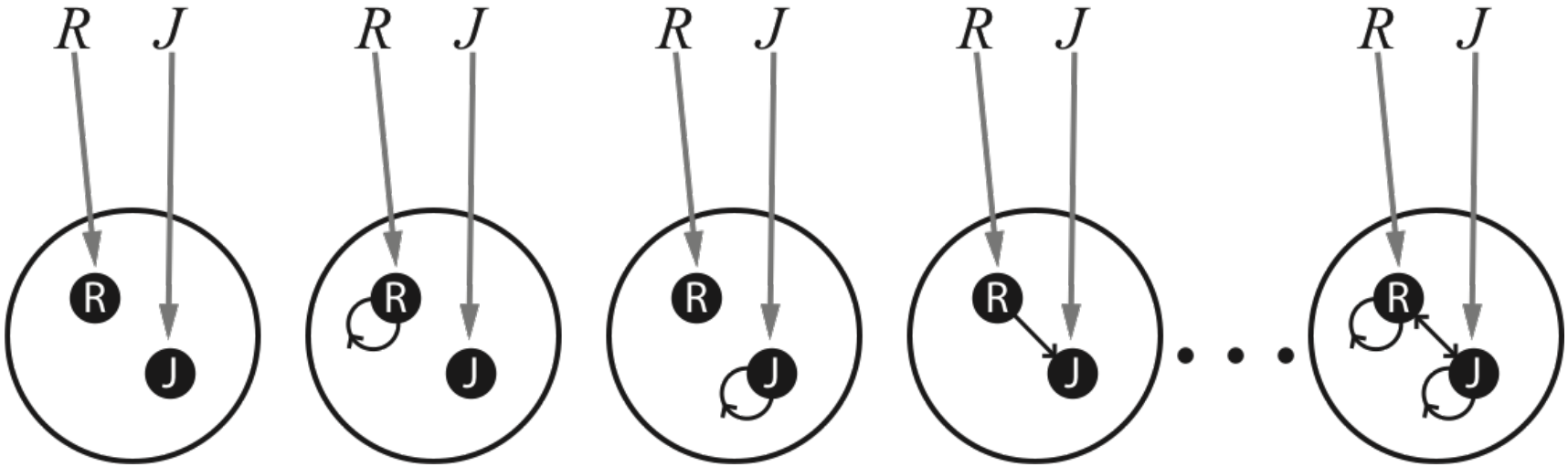
Indeterminate number of objects

- Let's look at all possible models for a language with
  - Two constants: R, J
  - One binary relation: B



"Database" semantics

- Typically conventions
  - Closed-world: Atomic sentences not in KB are false
  - Unique names: Different constants refer to different objects
  - Domain closure: Only allow model objects that are associated with constant symbols



Quantifiers

- Allow variables in addition to constants

$$Homework(x, 154)$$

- Sentences with free variables: $S(x)$
- Quantifiers bind free variables

$\forall x : S(x)$	is true if S(x) is true for all instantiations of x (i.e., for each possible object in the model)
$\exists x : S(x)$	is true if S(x) is true for at least one instantiation of x (i.e., for some object)

- Example:
  - All homeworks in 154 are hard

$\forall x : (Homework(x, 154) \Rightarrow Hard(x))$

  - At least one of the 154 homeworks is hard

$\exists x : Homework(x, 154) \wedge Hard(x)$

$\forall x : Homework(x, 154) \Rightarrow Hard(x)$

Properties of quantifiers

- Is $\forall x \ \forall y \ S(x, y)$ the same as $\forall y \ \forall x \ S(x, y)$?
- Is $\exists x \ \exists y \ S(x, y)$ the same as $\exists y \ \exists x \ S(x, y)$?
- Is $\exists x \ \forall y \ S(x, y)$ the same as $\forall y \ \exists x \ S(x, y)$?

$\exists x \forall y$ Loves(x, y) There is someone who loves everyone

$\forall y \exists x$ Loves(x, y) Everybody is loved by someone

De Morgan's law for quantifiers

- Each quantifier can be expressed by the other (they are *dual* to each other)

$$\neg \forall x \; S(x) \quad \equiv \quad \exists x \; \neg S(x)$$