CS 101

Numerical Geometric Integration

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Today's Show

Lagrangian/Hamiltonian mechanics

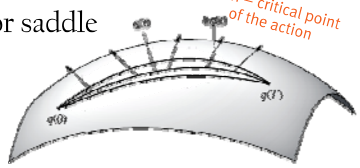
- continuing last lecture
  - pointing to more geometric characteristics
  - but still in continuous world for now

Reminder

Physical path **extremizes** action

$$\delta S(q) = \delta \int_0^T L(q, \dot{q})dt = 0$$

- min, max, or saddle



Non Conservative Systems

Lagrange-D'Alembert Principle

$$\delta \int_0^T L(q, \dot{q})dt + \int F(q, \dot{q}) \cdot \delta q \, dt = 0$$

- work created by the external forces
  - see "virtual work"
- forced Euler-Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] + F(q, \dot{q}) = 0$$

Generalized Momentum

Conjugate (or canonical) momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

- we've seen it last time for particles
- but applies for any generalized coord
  - angular momentum for instance!
- N vector for N-DOF system

Noether's Theorem (I)

Suppose invariance of Lagrangian

- $L(q, \dot{q}) = L(q^\varepsilon, \dot{q}^\varepsilon) \ \forall \varepsilon$
- by one-parameter family of paths $q^\varepsilon(t)$
  - with $q^0(t) = q(t)$

We define: $\xi(t) = \frac{\partial q^\varepsilon(t)}{\partial \varepsilon}|_{\varepsilon=0}$

Infinitesimal symmetry direction

Noether's Theorem (II)

Invariance of Lagrangian means…

- invariance of action

$$S(q) = \int_0^T L(q,\dot{q})dt$$

- momentum preserved in direction of ξ

$$0 = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \int_0^T L(q^\varepsilon, \dot{q}^\varepsilon)dt$$

$$= \frac{\partial L}{\partial \dot{q}}(q(T),\dot{q}(T)) \cdot \xi(T) - \frac{\partial L}{\partial \dot{q}}(q(0),\dot{q}(0)) \cdot \xi(0)$$

Noether's Theorem

Symmetries of a dynamical system give conserved quantities or conservation laws

- what's preserved: $\frac{\partial L}{\partial \dot{q}} \cdot \xi$

Reason behind tons of invariances

- even in quantum field theory!

Classical Examples

Translation Invariance

$$q^\varepsilon(t) = q(t) + \varepsilon T$$

- symmetry: $\xi(t) = T$
- so: 3xN vector!

$$\frac{\partial L}{\partial \dot{q}} \cdot T \quad \text{constant}$$

  - linear momentum preservation
    - T has 3 components, by the way

Classical Examples

Rotational Invariance (directional symmetry of space)

$$q^\varepsilon(t) = \exp(\varepsilon\Omega)q(t)$$

- antisymmetric matrix $\boldsymbol{\Omega} = \boldsymbol{\omega}^*$
  - or R(ε) $q(t)$ in2D if you prefer
- so:

$$\left(\frac{\partial L}{\partial \dot{q}} \times q\right).\omega \quad \text{constant}$$

  - angular momentum preservation

Energy

Another example of symmetry/invariance!

- systems not depend on time explicitly
  - laws of physics that don't change w/ time
- symmetry for $q^\varepsilon(t) = q(t+\varepsilon)$
  - in 4D: $\boldsymbol{\xi(t) = \frac{\partial}{\partial t}}$
- invariance of $\boldsymbol{\frac{\partial L}{\partial \dot{q}}(q,\dot{q}) \cdot \dot{q} - L(q,\dot{q})}$
  - say hello (again) to the Hamiltonian

Constrained Systems (I)

Let's keep it simple for now

- constrain configuration through

$$g(q(t)) = 0$$

- restrict motion within manifold
  - e.g., pendulum – restricted Lagrangian

Adapting variational principle:

$$\boldsymbol{\delta \int_0^T [L(q,\dot{q}) + \lambda(t) \cdot g(q(t))]dt = 0}$$

Constrained Systems (II)

With:

- λ = Lagrange multipliers
  - to enforce constraint
- think of it as force to stay on track
  - "tension" of string for pendulum
  - but no work done! (proof?)

How to Get this New Principle?

Augmented Lagrangian

- $\hat{L}(q, \dot{q}, \lambda, \dot{\lambda})$ (can also be done for Hamiltonian)

Take variations w.r.t. all variables

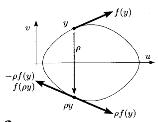
- variation w.r.t. λ: constraint
- variation w.r.t. q: Lagrange-D'Alembert!
  - but constraint forces no work

Btw, Noether's still applies

ρ-Reversible Systems

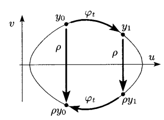
$\dot{y} = f(y)$ is *ρ-reversible* if:

$$\rho \circ f(y) = -f(\rho \circ y) \;\; \forall y$$



A map Φ(y) is *ρ-reversible* if:

$$\rho \circ \Phi = \Phi^{-1} \circ \rho$$



Time-Reversible Systems

$$\rho(u, v) = (u, -v)$$

$$\dot{u} = f(u, v), \dot{v} = g(u, v)$$

Example:
$$\begin{aligned} f(u, -v) &= -f(u, v) \\ g(u, -v) &= g(u, v) \end{aligned}$$

Special Cases:

$$\dot{u} = v, \dot{v} = g(u)$$

$$H(p, q) = H(-p, q)$$