1.) Lie group integrators: examples

2.) Nonholonomic integrators: basics

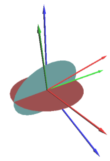
Geometric Integrators
Caltech, Feb 23, 2008

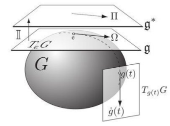
Geometric Integration on SO(3)

Consider a system evolving on the group of rotations $G = \mathrm{SO}(3)$

- configuration: matrix $R \in \mathrm{SO}(3)$
- body-fixed velocity $\Omega \in \mathfrak{so}(3)$ defined by $\Omega = R^T \dot{R}$
- intertia tensor $\mathbb{I} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$
- Lagrangian

$$L(R, \dot{R}) = \frac{1}{2} \langle R^T \dot{R}, \mathbb{I}(R^T \dot{R}) \rangle,$$





from D.Holm's textbook

- Example: write product using matrix trace

$$L(R, \dot{R}) = \frac{1}{2} \mathrm{tr}(\Lambda \dot{R}^T \dot{R}),$$

$\Lambda = \mathrm{diag}((-J_1 + J_2 + J_3)/2, (-J_2 + J_1 + J_3)/2, (-J_3 + J_1 + J_2)/2)$.
with J_1, J_2, J_3: principle moments of intertia

Discretization

- continuous curve $R : [0, T] \to SO(3) \Rightarrow$ set of points $\{R_0, ..., R_N\}$
- approximation using $R_k \approx R(kh)$, with time-step $h = T/N$
- assume constant velocity along each discrete segment

$$\dot{R}(t) \approx (R_{k+1} - R_k)/h, \quad t \in [kh, (k+1)h].$$

- Lagrangian approximated (for $t \in [kh, (k+1)h]$) according to

$$L(R(t), \dot{R}(t)) \approx \frac{1}{2} \mathrm{tr} \left(\Lambda \left(\frac{R_{k+1} - R_k}{h} \right)^T \left(\frac{R_{k+1} - R_k}{h} \right) \right).$$

Using properties of the trace this simplifies to

$$L(R(t), \dot{R}(t)) \approx -\frac{1}{h^2} \mathrm{tr} \left(\Lambda R_k^T R_{k+1} \right), \quad t \in [kh, (k+1)h].$$

Discrete variational principle

Moser-Veselov / Bobenko-Suris Lie group discrete approach

- Pontryagin-type principle (extra constraint $\Rightarrow$ relax variations) (Marsden, Pekarsky, Shkoller)
- The discrete trajectory must satisfy

$$\delta \sum_{k=0}^{N-1} -\frac{1}{h} \mathrm{tr}(\Lambda R_k^T R_{k+1}) + \sum_{k=0}^{N} \frac{1}{2h} \mathrm{tr}(\lambda_k (R_k^T R_k - \mathrm{I})) = 0,$$

where $\lambda_k \in \mathbb{R}^{3\times 3}$ is a matrix of multipliers enforcing orthogonality of R. Variations of R_k and λ_k are free.

- Note that

$$\mathrm{tr}(\lambda_k (R_k^T R_k - \mathrm{I})) = \mathrm{tr}(\lambda_k^T (R_k^T R_k - \mathrm{I}))$$

then a symmetric matrix λ_k is sufficient to enforce this condition.

Integrator derivation

Taking varitions with respect to each R_k

$$\mathrm{tr}(\Lambda R_{k-1}^T \delta R_k + \Lambda \delta R_k^T R_{k+1} - \lambda_k R_k^T \delta R_k) = 0,$$

or equivalently

$$\mathrm{tr}((\Lambda W_{k-1} + \Lambda W_k^T - \lambda_k)\eta_k) = 0, \qquad (1)$$

where $W_k = R_k^T R_{k+1}$ and $\eta_k = R_k^T \delta R_k$.

$$\lambda_k = \Lambda W_{k-1} + \Lambda W_k^T.$$

Since λ_k is symmetric

$$\Lambda W_{k-1} + \Lambda W_k^T = W_{k-1}^T \Lambda + W_k \Lambda,$$

and if we define $\mu_k = W_k \Lambda - \Lambda W_k^T$, this becomes equivalent to

$$\mu_k = W_{k-1}^T \mu_{k-1} W_{k-1}.$$

Since μ_k is skew-symmetric $\Rightarrow \mu_k \in \mathfrak{so}(3)^*$, this can be written as

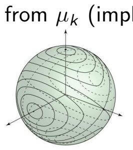
$$\mu_k = \mathrm{Ad}^*_{W_{k-1}} \mu_{k-1}.$$

General integrator equations

In summary, the equations of the integrator are

$$\mu_k = \mathrm{Ad}^*_{W_{k-1}} \mu_{k-1}, \quad \text{\% Momentum update: explicit} \qquad (2)$$
$$\mu_k = W_k \Lambda - \Lambda W_k^T, \quad \text{\% Legendre transform: implict} \qquad (3)$$
$$W_k^T W_k = \mathrm{I}, \quad \text{\% Orthogonality constraint: implict} \qquad (4)$$
$$R_{k+1} = R_k W_k. \quad \text{\% Rotation update: explicit} \qquad (5)$$

Given: μ_{k-1}, W_{k-1}, R_k
First update μ_k from μ_{k-1} and W_{k-1} (explicitly); then find W_k from μ_k (implicitly); update R_{k+1} from R_k and W_k (explicitly).



Can we accurately reproduce these periodic orbits?

The sphere with radius $\|\mu_0\|$

Direct Solution

- Solve for W in terms of μ directly using the matrix elements W^{ij} as uknowns. The equation $\mu = W\Lambda - \Lambda W^T$ is equivalent to requiring that

$$\begin{aligned} \mu_1 &= \Lambda_2 W^{32} - \Lambda_3 W^{23}, \\ \mu_2 &= \Lambda_3 W^{13} - \Lambda_1 W^{31}, \\ \mu_3 &= \Lambda_1 W^{21} - \Lambda_2 W^{12}, \end{aligned} \tag{6}$$

where

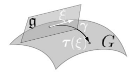
$$\mu = \begin{bmatrix} 0 & -\mu_3 & \mu_2 \\ \mu_3 & 0 & -\mu_1 \\ -\mu_2 & \mu_1 & 0 \end{bmatrix},$$

and Λ_i are the diagonal elements of the matrix Λ.

- 3 linear equations + 6 orthogonality conditions on W: 9 implicit equations $\Rightarrow$ polynomial roots / Newton method.

Parametrized Solution

Parametrize $W = \tau(\xi)$ using parameters ξ: e.g. exponential coordinates, or Cayley parameters



- Lie algebra identification $\mathfrak{so}(3) \sim \mathbb{R}^3$. Define $\widehat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$

$$\widehat{\omega} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{7}$$

$\mathfrak{so}(3)$ basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$, $\widehat{e}_i \in \mathfrak{so}(3)$ where $\{e_1, e_2, e_3\}$ is the standard $\mathbb{R}^3$-basis. Elements $\xi \in \mathfrak{so}(3)$ correspond to $\omega \in \mathbb{R}^3$ by $\xi = \omega^\alpha \widehat{e}_\alpha$, or $\xi = \widehat{\omega}$. Operator Ad becomes $\mathrm{Ad}_R\, \omega = R\omega$.

- Example: use the Cayley map

$$\mathrm{cay}(\widehat{\omega}) = \mathrm{I} + \frac{4}{4 + \|\omega\|^2}\left(\widehat{\omega} + \frac{\widehat{\omega}^2}{2}\right). \tag{8}$$

- In order to compute W given μ we solve

$$\mu = \mathrm{cay}(\widehat{\omega})\Lambda - \Lambda\, \mathrm{cay}(-\widehat{\omega})$$

for the 3 elements of ω (implicitly) and then find $W = \tau(\hat{\omega})$.

Explicit approximate solution

Use τ to represent W but trunacte its expression in the Legendre transform equation

- First order truncation of the exponential map $\exp(\xi) = \sum_{i=0}^{\infty} \frac{\xi^i}{i!}$:

$$\tau(\widehat{\omega}) = \exp(\widehat{\omega}) \approx \mathrm{I} + \widehat{\omega}.$$

- The Leg. equation reduces to $\mu = \widehat{\omega}\Lambda + \Lambda\widehat{\omega}$, and using standard inertia matrix $J = \mathrm{diag}(J_1, J_2, J_3)$ in identification $\mathfrak{so}(3) \sim \mathbb{R}^3$, the integrator is explicit:

$$\begin{aligned} \omega_k &= J^{-1} W_{k-1}^T J \omega_{k-1}, \\ W_k &= \exp(\omega_k), \\ R_{k+1} &= R_K W_k. \end{aligned}$$

Nonholonomic Dynamics

Systems with nonintegrable constraint on the velocities

- configuration space Q
- regular distribution $\mathcal{D}$: collection of subspaces $\mathcal{D}_q \subset T_q Q$
- Lagrangian $L : TQ \to \mathbb{R}$,
- control force $f : [0, T] \to T^*Q$.

For a curve $(q(t), v(t), p(t))$ in $TQ \oplus T^*Q$, $t \in [0, T]$ the d'Alembert-Pontryagin principle states that

$$\begin{aligned} &\delta \int_0^T [L(q, v) + \langle p, \dot{q} - v \rangle]\, dt + \int_0^T \langle f, \delta q \rangle dt = 0, \\ &\delta q \in \mathcal{D}_q \quad \text{and} \quad v_q \in \mathcal{D}_q, \end{aligned} \tag{9}$$

for variations that vanish at the endpoints.
Note: nonholonomic vs. vakonomic (~~$\delta q \in \mathcal{D}_q$~~)
symplectic structure not preserved in general

Continuous Equations of Motion

- Constraints $\mathcal{D}_q$ defined by m functions $\omega^a : TQ \to \mathbb{R}$, $a = 1, \ldots, m$ linear in the velocities and satisfy $\omega^a(q, \dot{q}) = 0$.
- After taking variations we get

$$\begin{aligned} \dot{q} &= v, \\ p &= \frac{\partial L}{\partial v}, \\ \langle \dot{p} &- \frac{\partial L}{\partial q} - f, \delta q \rangle = 0, \\ \omega^a(q) &\cdot v = 0. \end{aligned}$$

Allowed variations are such that $\omega^a(q) \cdot \delta q = 0$ and

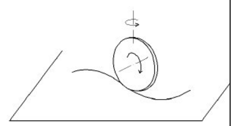
$$\dot{p} = \frac{\partial L}{\partial q} + f + f_{con},$$

where f_{con} are forces necessary to enforce constraints.

- $f_{con} = \lambda_a \omega^a(q)$ to cancel any acceleration in $\dot{p} - \frac{\partial L}{\partial q} - f$ not aligned with the constraints. λ_a are called Lagrangian multipliers denoting the magnitude of the constraint forces.

The vertical rolling disk

Configuration space $Q = SE(2) \times S^1$, with pose $(x, y, \phi) \in SE(2)$ and the rotation angle $\theta \in S^1$.



- Lagrangian

$$L(x, y, \phi, \theta, \dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} J\dot{\phi}^2,$$

m: mass, I, J: moments of intertia

- nonholonomic constraints are (with R: the disk radius)

$$\dot{x} = R(\cos\phi)\dot{\theta}, \qquad \dot{y} = R(\sin\phi)\dot{\theta}$$

or in the form $\omega^a \cdot (\dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = 0$

$$\omega^1 = (1, 0, 0, -R\cos\phi), \qquad \omega^2 = (0, 1, 0, -R\sin\phi).$$

- Controlled by torques u^ϕ and u^θ

The vertical rolling disk: continuous equations

The momentum is

$$p = \frac{\partial L}{\partial v} = (m\dot{x}, m\dot{y}, I\dot{\phi}, J\dot{\theta}).$$

The dynamics equation becomes

$$\begin{aligned} m\ddot{x} &= \lambda_1, \\ m\ddot{y} &= \lambda_2, \\ I\ddot{\phi} &= u^{\phi}, \\ J\ddot{\theta} &= -R\cos\phi\lambda_1 - R\sin\phi\lambda_2 + u^{\theta}. \end{aligned}$$

Differentiating the constraints and substituting

$$\begin{aligned} J\ddot{\phi} &= u^{\phi}, \\ (I + mR^2)\ddot{\theta} &= u^{\theta}, \end{aligned}$$

which along with the constraints determine the dynamics.

Nonholonomic Discrete Mechanics

- Discretization: represent $q : [0, T] \to Q$ by set $\{q_0, ..., q_N\}$
- Approximation: $q(kh) \approx q_k$, where $h = T/N$ is the time-step.
- Discrete d'Alembert-Pontryagin principle

$$\delta \sum_{k=0}^{N-1} [hL(q_{k+\alpha}, v_k) + \langle p_k, (q_{k+1} - q_k) - hv_k\rangle] + \sum_{k=0}^{N-1} h\langle f_{k+\alpha}, \delta q_{k+\alpha}\rangle = 0,$$

$$\delta q_k \in \mathcal{D}_{q_k} \quad \text{and} \quad v_k \in \mathcal{D}_{q_{k+\alpha}},$$

where $\alpha \in [0, 1]$: determines the interpolate quadrature point; notation: $x_{k+\alpha} := (1-\alpha)x_k + \alpha x_{k+1}$.

Discrete Equation of Motion

The resulting equations are

$$q_{k+1} = q_k + hv_k, \qquad \text{\% configuration update}$$

$$p_k = \frac{\partial L}{\partial v}(q_{k+\alpha}, v_k), \qquad \text{\% Legendre transform}$$

$$\frac{p_k - p_{k-1}}{h} \qquad \text{\% momentum update}$$

$$= (1-\alpha)\left(\frac{\partial L}{\partial q}(q_{k+\alpha}, v_k) + f_{k+\alpha}\right) + \alpha\left(\frac{\partial L}{\partial q}(q_{k-1+\alpha}, v_{k-1}) + f_{k-1+\alpha}\right)$$

$$+ (\lambda_a)_k \omega^a(q_k),$$

$$\omega^a(q_{k+\alpha}) \cdot v_k = 0. \qquad \text{\% velocity constraint}$$

Vertical Disk Integrator

Disk velocity $v = (v^x, v^y, v^{\phi}, v^{\theta})$. The discrete momentum is

$$p_k = \frac{\partial L}{\partial v}(q_{k+\alpha}, v_k) = (mv_k^x, mv_k^y, Iv_k^{\phi}, Jv_k^{\theta})$$

The discrete constraints are

$$v_k^x = R\cos(\phi_{k+\alpha})v_k^{\theta}, \qquad v_k^y = R\sin(\phi_{k+\alpha})v_k^{\theta}.$$

From the dynamics equation the multipliers can be computed as

$$\lambda_1 = m(v_k^x - v_{k-1}^x)/h, \qquad \lambda_1 = m(v_k^y - v_{k-1}^y)/h.$$

Substituting and simplifying the discrete dynamics becomes

$$I(v_k^{\phi} - v_{k-1}^{\phi})/h = (1-\alpha)u_{k-1+\alpha}^{\phi} + \alpha u_{k+\alpha}^{\phi}$$

$$\left[\left(J + mR^2\cos(\alpha h v_k^{\phi})\right)v_k^{\theta} - \left(J + mR^2\cos((1-\alpha)hv_k^{\phi})\right)v_{k-1}^{\theta}\right]/h$$

$$= (1-\alpha)u_{k-1+\alpha}^{\theta} + \alpha u_{k+\alpha}^{\theta}.$$

There is a lot more...

We've looked at only the most basic nonholonomic case. Things get more interesting when

- incorporating group symmetries (the principal bundle case)
- systmes with multiple bodies or internal joints
- studying relation to the continuous case (e.g. what is discrete connection, discrete curvature, discrete gyroscopic forces, how is momentum evolution different?)