Lie group integrators

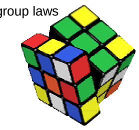
Marin Kobilarov

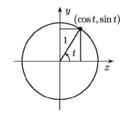
CS 101: Numerical Geometric Integration

February 17, 2009

Groups

- What is a group?
  - a *set* of elements and an *operation* that creates another element also in the group, i.e. it preserves the group
  - This fact is also associated with a *symmetry*, i.e. the group, or a property of the group is invariant to this operation
  - The set and its operation satisfy a number of basic group laws
- Group theory
  - Number theory, algebraic equations, and geometry
  - Studies algebraic and geometric structures
  - A fundamental tool in mathematics, e.g.:
    - Solving algebraic equations
    - Combinatorics, cryptography
    - Differential equations / manifolds
    - Quantum mechanics, string theory, etc…
- Lie Groups
  - Continuous groups with smooth structure
  - Continuous symmetries



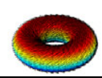


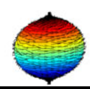
Origin of groups

- Évariste Galois Oct 25, 1811 -- May 31, 1832
  - First used the word "group" to represent a group of permutations
  - Galois theory: e.g. conditions whether a polynomial is solvable by radicals
  - Radical republican, documented some of his main ideas while in prison
  - Shot in the stomach and died after a duel, at age 20
- Sophus Lie Dec 17, 1842 – Feb 18, 1899
  - Extends Galois' work to continuous transformation groups *(Lie groups)*
  - Key idea: continuous groups can be studied by *linearizing* them and looking at the *infinitesimal* elements (vectors) that generate them; these generators satisfy a linearized version of a group law and form an *algebra (Lie algebra)*
- Other: Klein, Killing, Cartan, Weyl, Chevalley

Groups

- Group: a set G and operation · with the following properties:
  - Closure: for all a,b ∈ G, a· b ∈ G
  - Associativity: for all a,b,c ∈ G, (a· b)· c = a · (b · c) (order of operation)
  - Identity: there exists e ∈ G, such that e · a = a · e = a
  - Inverse: for each a ∈ G, there exists b ∈ G, such that a· b = b· a = e
- Lie Group: a continuous group G that is also a manifold
  - Locally looks like an Euclidean space
  - Multiplication and inversion must be smooth maps, i.e. small changes in the domain lead to small changes in the range
- Examples:
  - Groups: the integers under additions, the non-zero rationals under multiplication
  - Lie groups:
    - vectors in $\mathbf{R}^n$ under addition
    - The circle S^1 under angle addition mod 2π or complex number multiplication
    - The group of invertible matrices $GL_n(\mathbf{R})$ under matrix multiplication
    - The euclidean group SE(3) of rigid body transformations
    - Sphere S^3 of quaternions under quaternion multiplication
    - Not Lie groups: the spheres S^2, S^n, n>3 (they are not parallelizable, which causes loss of smoothness), the hairy ball theorem





Lie Groups

- Key Ideas
  - Smooth structure → study using its local or linearized version
  - Basis at the identity → Treat small infinitesimal group transformations as vectors attached at the identity which can locally span all directions of motion
  - Symmetry → once such a local basis is created (i.e. a basis of vectors of the tangent space at the identity, called T_eG) it can be transformed around to any point g∈G to create a local basis at that point spanning T_gG
  - Thus the whole group can be described in terms of the tangent space at the identity, or its *Lie algebra*
- Example: take $G=O^n$, where $O^n=\{R \mid R^T R = I_n\}$, the set of orthogonal matrices
  - Inifinitesimal group elements A are such that $I+\epsilon A \in G$ for some small ϵ:
    i.e. $(I+\epsilon A)^T (I+\epsilon A)=I \rightarrow A^T + A = 0$ ignoring ϵ^2
  - The Lie algebra is the set $\{A \mid A^T = -A\}$ of all skew-symmetric matrices
- Lie Algebra
  - A space of vectors at the group identity with an operation (bracket)

Lie Algebra

Key Idea: a linearized or infinitesimal version of a Lie group

² A space of vectors with a linear operation (bracket [¢¢ : g£ g ! g)

² [¢¢ is the linearized version of the group commutator $[g,h]=ghg^{-1}h^{-1}$, for g,h 2 G: i.e. what happens if one commutes group transformations?

² Satisfy the following properties:

– Anti-commutativity: for all x,y 2 g, $[x,y]=$ ¡ $[y,x]$ (comes from the commutator)

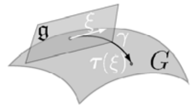
– Jacobi identity: for all x,y,z 2 g, $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ (linearized version of the group multiplication associativity property)

² Example: for matrix groups $[X,Y]=XY$ ¡ YX satisfies these properties

Lie Algebra Properties

Operations (might not be intuitive at first)

- The exponential map $\exp : \mathfrak{g} \to G$, defined by $\exp(\xi) = \gamma(1)$, with $\gamma : \mathbb{R} \to G$ is the integral curve through the identity of the left invariant vector field associated with $\xi \in \mathfrak{g}$ (hence, with $\dot{\gamma}(0) = \xi$);
- Conjugation map $\mathrm{I}_g : G \to G$, $h \to ghg^{-1}$, for $g, h \in G$ (think: similarity transformation)
- Adjoint map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$: the tangent of the conjugation (think: change of basis of vectors)
- Bracket operator $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$: $\mathrm{ad}_x\, y = [x, y]$ is the linearized version of the Ad operator: $\mathrm{ad}_x\, y = \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\exp(tx)}\, y$



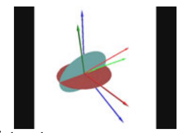
Example: rigid body rotations

- Consider a rigid body rotating around a fixed point
- Fix a frame of three orthonormal vectors at that point
- This frame rotates together with the body
- It forms the group of special orthogonal 3x3 matrices $SO(3) = \{ R \in O(3) | \det(R) = 1 \}$
- the Lie algebra of skew-symmetric matrices is denoted $\mathfrak{so}(3) = \{ \xi \in \mathbf{R}^{3\times3} | \xi^T = -\xi \}$

  These elements $\xi \in \mathfrak{so}(3)$ can be expressed using a vector $\omega \in \mathbb{R}^3$ through

$$\xi = \widehat{\omega} = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix},$$

  where ω plays the role of the *body-fixed* angular velocity. Since the $\widehat{\cdot}$ map is isomorphism, the Lie algebra can be identified with $\mathbb{R}^3$:



- Exponential map

$$\exp(\omega) = \begin{cases} \mathbf{I}_3, & \text{if } \omega = 0 \\ \mathbf{I}_3 + \frac{\sin\|\omega\|}{\|\omega\|}\widehat{\omega} + \frac{1-\cos\|\omega\|}{\|\omega\|^2}\widehat{\omega}^2, & \text{if } \omega \neq 0 \end{cases},$$

- Adjoint map $\mathrm{Ad}_R\, \omega = R\omega$: changes from body-fixed to inertial-frame velocity
- Bracket is the cross-product $[\omega, v] = \omega \times v$

Riemannian Metric on a Lie group

Inner product (metric)

- generalization of vector dot product
- a function $\langle\langle \cdot, \cdot \rangle\rangle : V \times V \to \mathbb{R}$ on vector space V
- Properties: symmetry; linearity, positive-definiteness

Left-invariant Riemannian metrics

- On a Lie group we can define a metric $\langle\langle \cdot, \cdot \rangle\rangle_g : T_gG \times T_gG \to \mathbb{R}$ on every tangent space T_gG by $\langle\langle v, w \rangle\rangle_g$, for some $v, w \in T_gG$
- Some metrics (left-invariant) are preseved after a left group transformation of the agrument vectors, i.e. $\langle\langle v, w \rangle\rangle_g = \langle\langle hv, hw \rangle\rangle_{hg}$ for some $h \in G$.
- this enables one to define only a single metric, i.e. at the identity $\langle\langle \cdot, \cdot \rangle\rangle$ and use it to compute the metric anywhere on the group by left-translating the argument vectors to the identity: $\langle\langle v, w \rangle\rangle_g = \langle\langle g^{-1}v, g^{-1}w \rangle\rangle$.
- Or, since $g^{-1}v \in T_eG \sim \mathfrak{g}$, by defining a metric on the Lie algebra

Mechanics and Variational Principles

One more thing: each tangent space T_gG has a dual: the linear space of elements which can be multiplied by elements in T_gG to give a real number

- it is denoted $T_g^*G = \mathrm{span}\{e^i | \langle e^i, e_j \rangle = \delta_{i,j}\}$, $\{e^i\}$ is the basis of T_g^*G while $\{e_i\}$ is the basis of T_gG and the *pairing* $\langle \cdot, \cdot \rangle$ is a function giving the number (think: dot product of vectors)

Metrics in mechanics

- A metric of fundamental importance is the *kinetic energy* $KE(\dot{g}) = \langle\langle \dot{g}, \dot{g} \rangle\rangle_g = \langle \mathbb{I}\xi, \xi \rangle$, for $\xi = g^{-1}\dot{g} \in \mathfrak{g}$
- Rigid body example: $KE = (\mathbb{I}\omega)^T\omega$

Variational principles

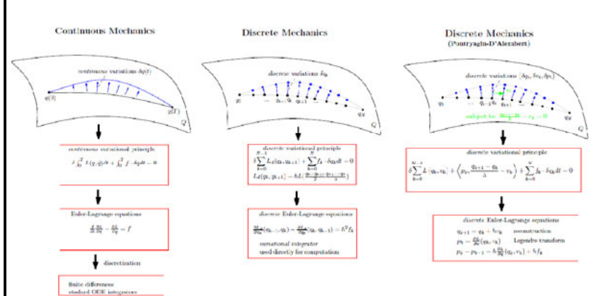
- Appears in the Lagrangian $L = KE(\dot{g})$: the basis for the underlying variational principle, i.e.

$$\delta \int L(g, \dot{g})dt = 0 \quad \Rightarrow \quad \left\langle \frac{d}{dt}\frac{\partial L}{\partial \dot{g}} - \frac{\partial L}{\partial g}, \delta g \right\rangle = 0$$

- Better to use $\xi = g^{-1}\dot{g}$: evolves in a linear space and decouples dynamics

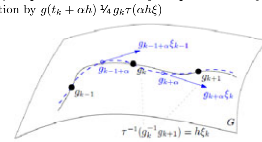
$$\delta \int \ell(g, \xi)dt = 0, \text{ subject to } \xi = g^{-1}\dot{g},$$

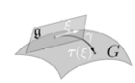
Review: variational principles and integrators



Trajectory discretization on Lie groups

- In the discrete setting the continuous curves $g(t), \xi(t)$, for $t \in [0, T]$ are approximated by a discrete set of points at equally spaced time intervals: $g : [0, T] \to G$ is given the temporal discretization $g^d = \{g_0, g_1, \ldots, g_N\}$ with $g_k := g(kh)$, where $h = T/N$ is the time step.
- the trajectory between discrete points is represented by choosing: a velocity $\xi_k \in \mathfrak{g}$ and a *retraction* map $\tau : \mathfrak{g} \to G$ which generates the approximation by $g(t_k + \alpha h) \approx g_k\tau(\alpha h\xi)$





Examples of maps τ generating the group locally (for matrix groups)

- *Exponential map*: $\exp : \mathfrak{g} \to G$, $\exp(\xi) = \sum_{i=0}^{\infty} \frac{\xi^i}{i!}$.
- *Cayley map*: $\mathrm{cay} : \mathfrak{g} \to G$, $\mathrm{cay}(\xi) = (e - \xi/2)^{-1}(e + \xi/2)$.

Pontryagin's principle on Lie groups

The Pontryagin viewpoint:
enforce the group constraint with a multiplier $\mu \in \mathfrak{g}^*$ in the Lie algebra dual.

Continuous

$$\delta \int_0^T [\ell(\xi) + \langle \mu, g^{-1}\dot{g} - \xi \rangle] + \int \langle f, g^{-1}\delta g \rangle = 0$$

$$\delta(g^{-1}\dot{g}) = \dot{\eta} + \mathrm{ad}_\xi\, \eta,\ \eta = g^{-1}\delta g$$

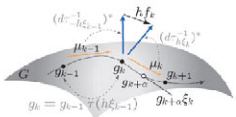
$$\mu = \frac{\partial \ell}{\partial \xi}$$
$$\dot{\mu} = \mathrm{ad}^*_\xi\, \mu + f$$

Discrete

$$\sum_{k=0}^{N-1} h\ell(\xi_k) + \langle \mu_k, \tau^{-1}(g_k^{-1} g_{k+1})/h - \xi_k \rangle + \sum_{k=0}^{N} \langle f_k, g_k^{-1}\delta g_k \rangle = 0$$

$$\delta(\tau^{-1}(g_k^{-1} g_{k+1})) = \mathrm{d}\tau^{-1}_{h\xi_k}(-\eta_k + \mathrm{Ad}_{g_k^{-1} g_{k+1}}\, \eta_{k+1})$$

$$\mu_k = \frac{\partial \ell}{\partial \xi}(\xi_k)$$
$$(\mathrm{d}\tau^{-1}_{h\xi_k})^* \mu_k - (\mathrm{d}\tau^{-1}_{-h\xi_{k-1}})^* \mu_{k-1} = h f_k$$



$$\mathrm{d}\tau_\xi^{-1} = \mathrm{I} - \frac{1}{2}\mathrm{ad}_\xi + \frac{1}{12}\mathrm{ad}_\xi^2 + \dots$$

reflects the "non-flatness" of the space

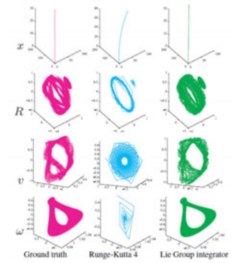
Example: rigid body rotation

Continuous

$$\Pi = \mathbf{I}\,\omega$$
$$\dot{\Pi} = \Pi \times \omega + f$$
$$\dot{R} = R\widehat{\omega}$$

Discrete (Lie-Trapezoid scheme)

$$\Pi_k = \mathbf{I}\,\omega_k$$
$$(\mathrm{I} - \tfrac{h}{2}\widehat{\omega_k})^T \Pi_k - (\mathrm{I} + \tfrac{h}{2}\widehat{\omega_{k-1}})^T \Pi_{k-1} = h f_k$$
$$R_{k+1} = R_k \tau(h\widehat{\omega})$$



Many other possible integrators

Other Lie group integrators

- based on different group/algebra discretization
- based on evolution of group elements only (the standard Veselov / Bobenko,Suris approach)
- not based on variational principles (e.g. standard RK in the algebra + group lifts)
- based on other geometric notions (e.g. plastic impact onto the group constraint surface)

Integrators for more complex systems

- multi-body systems / systems with internal shape
- systems with nonholonomic constraints (non-integrable velocity constraints)

More general *Lie grupoid* integrators

- capture complex structures through a more general approach
- i.e. can study complex structures by defining more abstract operations
- could be useful for easier derivation / implementation

References

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