CS 101.3: Numerical Geometric Integration

Homework Assignment #7
Due date: March 16th 2009 at 5:00pm.
All code should be submitted by email.
The written parts may be submitted by email
or in the clip outside Mathieu's office.

Abstract

In this assignment you will explore some of the results from backward error analysis and time adaption. If you have questions email patrickm@cs.caltech.edu - I will be happy to meet upon request.

1 Theory Part

1.1 Bounds for the Two-Body Problem

The two-body problem in two dimensions with normalized masses can be written as a Hamiltonian system for the second mass:

$$H(p,q) = \frac{1}{2}p^2 - \frac{1}{||q||}$$

We will consider starting in a region bounded by $||p|| \leq 2$ and $0.8 \leq ||q|| \leq 1.2$ and would like to integrate this with a symplectic RK method and get (conservatively) guaranteed good energy behavior. As always feel free to use Mathematica to help.

- Write the equations of motion in the standard $\dot{y} = f(y)$ notation.
- Find reasonable constants M and R such that for y_0 in the region we're considering then over the ball $||y - y_0|| \leq 2R$, $f(y)$ is complex analytic and satisfies $||f(y)|| \leq M$ (note these were the assumptions we made in class).
- Use the conditions of theorem 7.6 from the book (reproduced on the slides) to find h_0 and N taking half of the largest timestep allowed by the theorem. For this h, and assuming $\gamma = 30.5$, compute the bound on the distance between one step of the method and the solution to the N-truncated modified equation at that time.
- Using these values, for how many time steps can we safely guarantee the Hamiltonian of the original system will be bounded by a constant?

1.2 Bounds for the Time-Adapted Two-Body Problem

For the same two-body problem as above, consider the time reparameterization $dt/d\tau = \sigma(p, q)$ for a fictitious time variable τ, using $\sigma(p, q) = |q|^2$. Write the *modified Hamiltonian* $\widehat{H}$ discussed in slide 12 of Lecture 13, and derive the new equation of motion $y^{'} = J^{-1}\nabla\widehat{H}$. Now do the exact same analysis as above. Discuss how time reparameterization has affected the integrator.

2 Implementation Part

2.1 Computing Modified Equations

- Generate the expansion of the modified equation for the problem $\dot{y} = y^2$ using explicit Euler, implicit Euler, and implicit midpoint. Take the expansions up to the h^8 term. For explicit Euler and implicit midpoint you should get the results given in the slides for the early terms. You may (or may not) find the following Mathematica functions useful: D[], Series[], Collect[], Coefficient[]. The approach your code should take is the following:
  - Expand the method as a function of y_k to enough terms using $f(y) = y^2$. Note for implicit methods you will need to recursively substitute the definition of y_{k+1} until enough terms are only functions of y_k. Parts of the high-order terms can appear quickly in this process, so start truncating early or the computation will get out of hand. One nice trick for truncation is to use the Series[] function to expand your equation only up to the terms you care about and it will discard the others.
  - Using the expansion of your method, you can now compute the terms of the modified equation in order (since each term depends on the previous terms). Start with your current estimate of the modified equation and take the necessary products of it and its derivatives to get the terms needed for the computation of the next f_i. Once you have this use your new (better) estimate of the modified equation and repeat.
- Now for each method run the method on the problem (with fairly large h) and plot the points along with the exact solutions to the corresponding modified equation truncated after 1, 2, and 3 terms. For the "exact" solution of the modified equation you may either use Mathematica to solve for it algebraically or use a numerical method with very small time steps.

2.2 Time-Adapted Two-body Problem

Derive an adaptive-time integrator for the two-body problem of Section 1, using $\sigma(p, q) = |q|^2$ once again. This time, however, derive the integrator using the Lagrangian approach, i.e., using the traditional variational derivation where the time step $t_{k+1} - t_k$ is enforced to be $h\sigma(q_k, q_{k+1})$ through a Lagrange multiplier λ_{k+1} (slide 16 of Lecture 13). Try your integrator for various initial conditions and various values of h. Plot λ_k to see how the energy of the system is well/badly preserved.