CS 101.3: Numerical Geometric Integration

Homework Assignment #6
Due date: March 2nd 2009 at the beginning of class.
All code should be submitted by email.

Abstract

In this assignment you will derive a higher-order variational integrator and implement a Lie group integrator. Please note that the honor code applies: do the derivations yourself. If you have questions email patrickm@cs.caltech.edu - I will be happy to meet upon request.

1 Theory Part

1.1 High Order Variational Integrator

From a continuous Lagrangian $\mathcal{L}(q, \dot{q})$, we define the discrete Galerkin Lagrangian L (mentioned in class) as:

$$L(q_k, q_{k+1}, h) = \underset{Q_1,\ldots,Q_{s-1}}{\text{ext}} \; L^{\text{full}}(q_k, Q_1, \ldots, Q_{s-1}, q_{k+1}, h) \tag{1}$$

with:

$$L^{\text{full}}(Q_0, Q_1, \ldots, Q_s, h) = \sum_{i=1}^{i=s} w_i \, \mathcal{L}\left(\sum_{j=0}^{j=s} \phi_j(\alpha_i) Q_j, \frac{1}{h} \sum_{j=0}^{j=s} \dot{\phi}_j(\alpha_i) Q_j\right),$$

where $(w_i)_{i=1..s}$ are quadrature weights associated with quadrature points $0 \leq \alpha_1 < \ldots < \alpha_s \leq 1$ of a quadrature method of order at least s ($\int_0^1 f \sim \sum_i w_i f(\alpha_i)$), and $(\phi_j)_{j=0..s}$ are Lagrange basis functions of order s from $[0, 1]$ to $\mathbb{R}$ such that $\phi_j(\beta_i) = \delta_{ij}$, i.e., a basis of order-s polynomials.

- Choosing equally-spaced control times $\beta_0 = 0, \beta_1 = \frac{1}{3}, \beta_2 = \frac{2}{3}, \beta_3 = 1$, write out the Lagrange basis functions ϕ_j (feel free to use Mathematica to derive them).
- The 3-point Lobatto quadrature corresponds to $\alpha_1 = 0, \alpha_2 = 1/2, \alpha_3 = 1$ and $w_1 = w_3 = 1/6, w_2 = 4/6$ (it should remind you of Simpson's rule). Using this quadrature deduce an implicit expression of $L(q_k, q_{k+1}, h)$ based on the

extremization with respect to the additional points Q_1, Q_2 in Eq. 1 for $s = 3$. The expression should be in terms of $D_1\mathcal{L}$ and $D_2\mathcal{L}$ along with your quadrature points, control points and basis functions.

- Using this discrete Lagrangian, derive a variational integrator based on the Lobatto quadrature. Write out the equations that need to be solved, and point out which are solved together and for what variables.

1.2 Discrete Mosler-Veselov Integration

Consider the dynamics of a rigid body only undergoing rotations (i.e., the evolution is purely on $SO(3)$, generated by the Lagrangian $\mathcal{L}(R, \dot{R}) = \frac{1}{2}\text{tr}((R^{-1}\dot{R})^T \Lambda (R^{-1}\dot{R}))$ with Λ a symmetric matrix based on moments of inertia as defined in class. In the discrete setting, we call R_k the rotation at time t_k.

- By discretizing
$$R^{-1}\dot{R} \equiv R_{k+1}^T \frac{R_{k+1} - R_k}{h},$$
provide an expression of the discrete Lagrangian $L(R_k, R_{k+1})$.

- Use $R^{-1}\dot{R} \equiv R_k^T \frac{R_{k+1}-R_k}{h}$ instead. How does it affect the result?

- Use the properties of the trace and of rotation matrices to show that this last expression is equivalent to (up to a constant):
$$L(R_k, R_{k+1}) = -\frac{1}{h}\text{tr}(R_k \Lambda R_{k+1}^T)$$

- Now write the augmented action (with λ_k a matrix Lagrange multiplier) as:
$$S = \frac{1}{h}\left[-\sum_k \text{tr}(R_k \Lambda R_{k+1}^T) + \sum_k \frac{1}{2}\text{tr}(\lambda_k(R_k R_k^T - I))\right].$$
Extremizing this action considering an unconstrained rotation R_k yields:
$$R_{k+1}\Lambda + R_{k-1}\Lambda = \lambda_k R_k.$$
From this last update equation, deduce that the quantity: $m_k = R_{k+1}\Lambda R_k^T - R_k \Lambda R_{k+1}^T$ is preserved due to the fact that $\lambda_k = \lambda_k^T$. Note that this is the discrete angular momentum.

- Find how m_k and the μ_k (given in lecture 12) are related. Note that μ_k is the *body-fixed* angular momentum (also called the body angular momentum, or the angular momentum w.r.t. the body).

- Infer the discrete spatial angular velocity from the discrete body-fixed angular velocity W_k given in class.

2 Implementation Part

Implement the dynamics on SO(3) (for say, moments of inertia J_1, J_2, J_3 equal to $1, 2$, and 3 respectively) discussed in class, using a "direct" or "parameterized" solution. Find a way to deal with initial conditions, and verify numerically that the spatial momentum is preserved. You can plot the body angular momentum as points on a sphere, and should be able to observe the orbits (for varying initial conditions) depicted in class.