CS 101.3: Numerical Geometric Integration

Homework Assignment #3
Due date: Feb 9th 2009 at the beginning of class.
All code should be submitted by email.

Abstract

In this assignment, you are asked to derive various variational integrators, and to verify their symplecticity in a visual manner. Please note that the honor code applies: do the derivations yourself. If you have questions email patrickm@cs.caltech.edu - I will be happy to meet upon request.

1 Theory Part

One-Point Quadratures for Separable Lagrangian

Suppose that you are given a Lagrangian of the form: $L(q, \dot{q}) = K(\dot{q}) - W(q)$ where K is the kinetic energy $K(\dot{q}) = \frac{1}{2}\dot{q}^t M \dot{q}$, and W is a potential.

- For the quadrature of $\int_{t_k}^{t_{k+1}} W dt$, use a one-point quadrature based on the value of q at t_k, *i.e.*, based on q_k. Derive a variational integrator from that quadrature (write down the update rule linking q_{k-1}, q_k, q_{k+1}, and W).
- Derive another one-point quadrature using, this time, t_{k+1} as the only sample in time to approximate the integral of W between t_k and t_{k+1}. Derive the associated variational integrator.

Three-Point Quadrature for Separable Lagrangian

Recalling that the Simpson's rule for approximating integrals is:

$$\int_a^b f(x)\, dx \approx \left[\frac{1}{6}f(a) + \frac{4}{6}f(\frac{a+b}{2}) + \frac{1}{6}f(b)\right]\ (b-a),$$

derive a variational integrator that uses Simpson's rule for the discretization of the potential integrated in time.

2 Implementation part

Visualizing Symplecticity

The goal here is to look at the volume change in phase space of a few integrators with our favorite system, the pendulum. As we've discussed in class, this volume preservation is with respect to a *region of initial conditions* and how all of these points are carried in the flow over time. To visualize this you should take N initial conditions sampling some disk in phase space and then visualize all of these points plotted together at various time samples. Make N large enough to get a good approximation of the shape and volume without requiring excessive amounts of computation (use your own judgement). Similarly, for time step size and durations please find reasonable values so you can see what is happening with the various integrators (too small timesteps results in everything looks pretty good, while too large timesteps may change too fast to see what's happening). The experience of playing with these to find good values and see what happens as you change parameters is as much a part of the assignment as the final results you submit.

1. Implement the pendulum using the Störmer-Verlet method. Start with a sampling of initial conditions in some disk which is completely inside the set of points which will make the pendulum oscillate (but not containing $(0, 0)$). Make a few plots at various times (including $t = 0$) of these points to see how the shape and volume changes over time.

2. Do the same thing, only now have your initial conditions lie in a disk that is approximately half inside and half outside the region of oscillation. In other words, about half of your initial conditions should be such that $\dot{\theta}$ never changes sign.

3. Do the above two steps for implicit Euler, explicit Euler, and your variational integrator based on Simpson's rule.

4. (Optional, bonus) Make some nice animations of the above showing the various behaviors. Feel free to save these to .avi files (see Export[] if using Mathematica) if they take a while to generate. Note that if you're using a lot of points and the simulations are taking some time for each frame you can store each frame in a list and then ListAnimate the result after you're done to get smooth animations.

5. (Optional, bonus) Do the above for some more interesting/illustrative/fun initial shapes than a disk. Cats, bunnies, teapots, images, etc. are all welcome.