CS 101.3: Numerical Geometric Integration

Homework Assignment #2
Due date: Feb 2nd 2009 at the beginning of class.

Abstract

In this assignment, you are asked to expand on some of the derivations shown in class, and to implement a Lotka-Volterra model. Please note that the honor code applies: do the derivations yourself. If you have questions email patrickm@cs.caltech.edu - I will be happy to meet upon request.

1 Theory Part

1.1 Stability Conditions

- Using von Neumann method, derive a necessary condition for stability of "Lax method" for PDEs introduced in Lecture 2, *i.e.*, for the update:

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{v\,\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

- Same deal for the Lax-Wendroff method, *i.e.*, for the update:

$$u_j^{n+1} = u_j^n - \frac{C}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{C^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

  where $C := \frac{v\,\Delta t}{\Delta x}$ (sometimes called the convection number).

- Same deal for the "upwind method", *i.e.*, for (assuming $v > 0$):

$$u_j^{n+1} = u_j^n - \frac{v\,\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

1.2 Modified Differential Equations

We showed in class that Lax method is, in fact, integrating the PDE:

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 u}{\partial x^2}.$$

This was found by simply rearranging the terms of the update.

A more direct approach to find such "modified" equations is to take the update rule for u_j^{n+1}, and simply apply (spatial and temporal) Taylor expansions about u_j^n.

1. Try this approach to get the modified equation for the upwind method mentioned above; disregard the terms in $\Delta t^2, \Delta x^2$ and above in the result.
2. Differentiate the previous result by t, and by x. Combine these two resulting equations to get:
$$u_{tt} = v^2 u_{xx} + \mathcal{O}(\Delta t).$$
3. Use this last result in the equation you got in (1) to get a modified equation similar to Lax's. What happens when $v = \Delta x/\Delta t$?
4. Do the same treatment for the Lax-Wendroff method. Compare the high order term to the upwind's one.

1.3 Time Reversibility

Show that the condition $H(p,q) = H(-p,q)$ on the Hamiltonian of a system implies time reversibility of the flow. Also, work out the necessary conditions on the b_j's for a RK method to be time-reversible.

2 Implementation part

Lotka-Volterra Model and Discrete Gradient Methods

The Lotka-Volterra model can be used for modeling predator-prey relationships, assuming exponential growth of the prey population and exponential decay of the predator population in isolation. If we assume two species with populations u and v (u being the predator), the equations for the evolution of the system over time become:

$$\dot{u} = u(v - C_1), \quad \dot{v} = v(C_2 - u)$$

with C_1 and C_2 constants determining the rate of decay and growth of the isolated predator and prey respectively. It can be shown that this system has the invariant

$$H(u,v) = C_2 \ln u - u + C_1 \ln v - v$$

With this observation we can now rewrite our system of equations as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = uvJ^{-1}\nabla H(u,v) = B(u,v)\nabla H(u,v)$$

with $B(u,v) = uvJ^{-1}$ an antisymmetric matrix, and this invariant H should therefore be preserved by discrete gradient methods.

1. Taking $C_1 = 2$ and $C_2 = 1$ and initial conditions $u(0) = v(0) = 3$, integrate the Lotka-Volterra system using explicit Euler with timestep .1 and with implicit midpoint using timesteps .1 and .5. Plot the resulting trajectory in (u, v) space and plot H as a function of time. Run all your simulations to approximately time 30.

2. Implement the discrete gradient method using the midpoint discrete gradient given in class with timesteps .1 and .5 and produce the same plots. Evaluate B at the midpoint to keep the method symmetric.

   Note: The midpoint discrete gradient as given is degenerate when evaluated at $\bar{\nabla} H(y, y)$, i.e. at the same point (the condition $\bar{\nabla} H(y, y) = \nabla H(y)$ is true in the sense that the limit of $\bar{\nabla} H(y, \hat{y})$ as $\hat{y} \to y$ is $\nabla H(y)$, and this is sufficient - we never used this property in the proof of conservation, it is only for consistency). However, this means when giving your initial guess to FindRoot you should perturb the guess from the previous step (try by $\{.01, .01\}$) so that you don't get a complaint about division by zero.

3. Make 2 plots: one of trajectories of your discrete gradient method in u, v space with timestep of .2 with initial conditions $\{2.8, 2.8\}$, $\{3, 3\}$, and $\{3.2, 3.2\}$ on top of each other, and another with the same conditions only using B evaluated explicitly in time. Note this explicit evaluation will break the symmetry of the method, but not the conservation property.