Online Optimization in X-Armed Bandits

CS101.2

January 20th, 2009

Paper by S. Bubeck, R. Munos, G. Stoltz, C. Szepersvári

Slides by C. Chang

Review of Bandits

- Started with *k* arms
  - Integral, finite domain of arms
  - General idea: Keep track of average and confidence for each arm
  - Expected regret using $UCB_1 = O(\log n)$

Review of Bandits

- Last week
- Bandit arms against "adversaries"
  - Oblivious
    - $O(n^{2/3})$
  - Adaptive
    - $O(n^{3/4})$

Extending the Arms

- What about infinitely many arms?
- Draw arms from $X = [0, 1]^D$
  - D-dimensional vector of values from 0 to 1
- Mean-payoff function, *f*, maps from *X* → **R**
- No adversaries (fixed payoffs)

Extending the Arms

- What if there are no restrictions on the shape of *f*?

Extending the Arms

- What if there are no restrictions on the shape of *f*?
- Then we don't know anything about arms we haven't pulled

Extending the Arms

- What if there are no restrictions on the shape of *f*?
- Then we don't know anything about arms we haven't pulled
- With infinitely many arms, this means we can't do anything!

Extending the Arms

- Okay, so no continuity at all goes too far
- Generalize the mean-payoff function function to be “pretty smooth”
- That way, we can (hopefully) get information about a neighborhood of arms from a single pull
- We will use Lipschitz continuity

Lipschitz Continuity

- Intuitively, the slope of the function is bounded
- That is, it never increases or decreases faster than a certain rate
- This seems like it can give us information about an area with a single pull

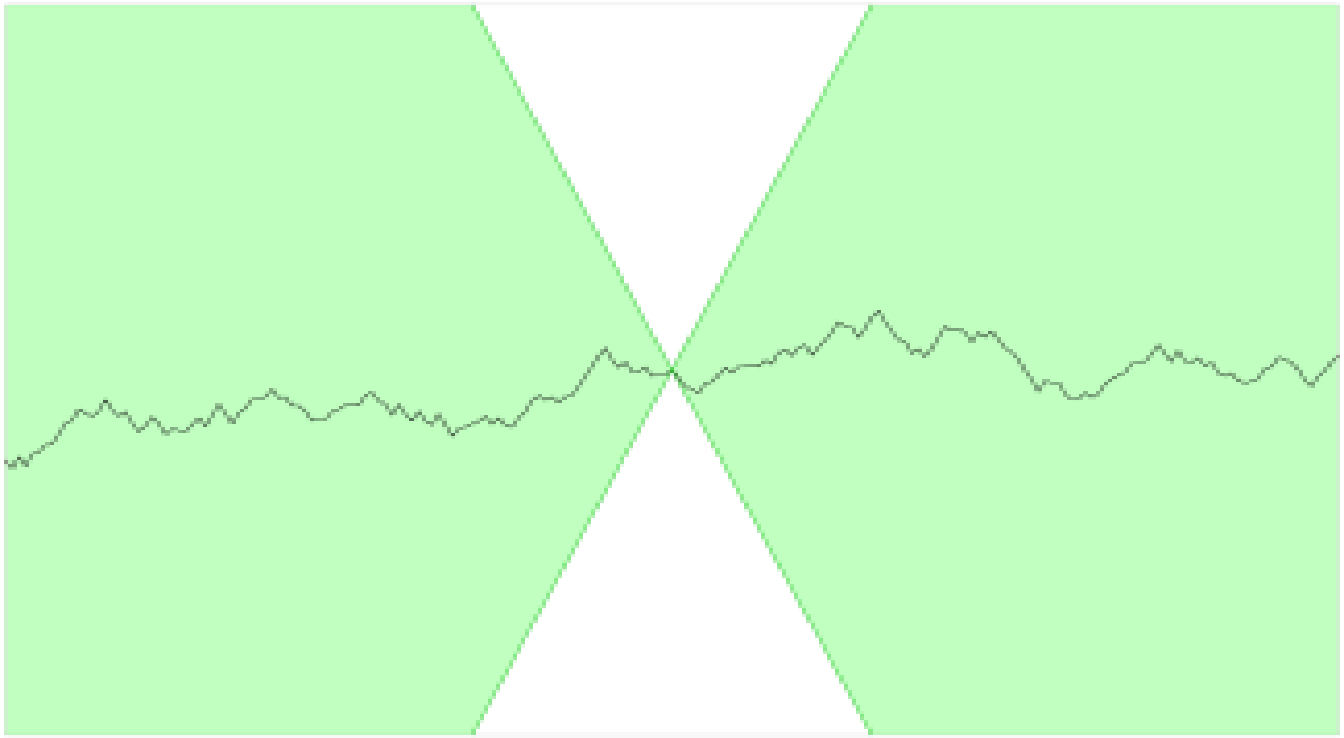
Lipschitz Continuity

- Formal definition:
- Function $f(x)$ is Lipschitz continuous if,
- Given a dissimilarity function, $d(x,y)$,
- $f(x) - f(y) \leq k \times d(x,y)$
- k is the Lipschitz constant

Lipschitz Continuity

- For a function *f* with a certain constant *k*, we call the function *k*-Lipschitz
- We'll assume 1-Lipschitz
  - For another *k*, we can just adjust the payoffs to make the function 1-Lipschitz
  - We're really just concerned with relative performance versus other strategies on the same *f*

Lipschitz Continuity



Function will stay inside the green cone

(Graphic taken with permission from Wikipedia under GNU Free Documentation License 1.2)

Lipschitz Functions

- Examples of functions that are Lipschitz:

Lipschitz Functions

- Examples of functions that are Lipschitz:
  - $f(x) = \sin(x)$
  - $f(x) = |x|$
  - $f(x,y) = x + y$

Lipschitz Functions

- Examples of functions that are Lipschitz:
  - $f(x) = \sin(x)$
  - $f(x) = |x|$
  - $f(x,y) = x + y$
- And functions that aren't:

Lipschitz Functions

- Examples of functions that are Lipschitz:
  - $f(x) = \sin(x)$
  - $f(x) = |x|$
  - $f(x,y) = x + y$
- And functions that aren't:
  - $f(x) = x^2$
  - $f(x) = x / (x - 3)$

Application

- Why would we need a bandit arm strategy for non-linear mean-payoff functions?

Application

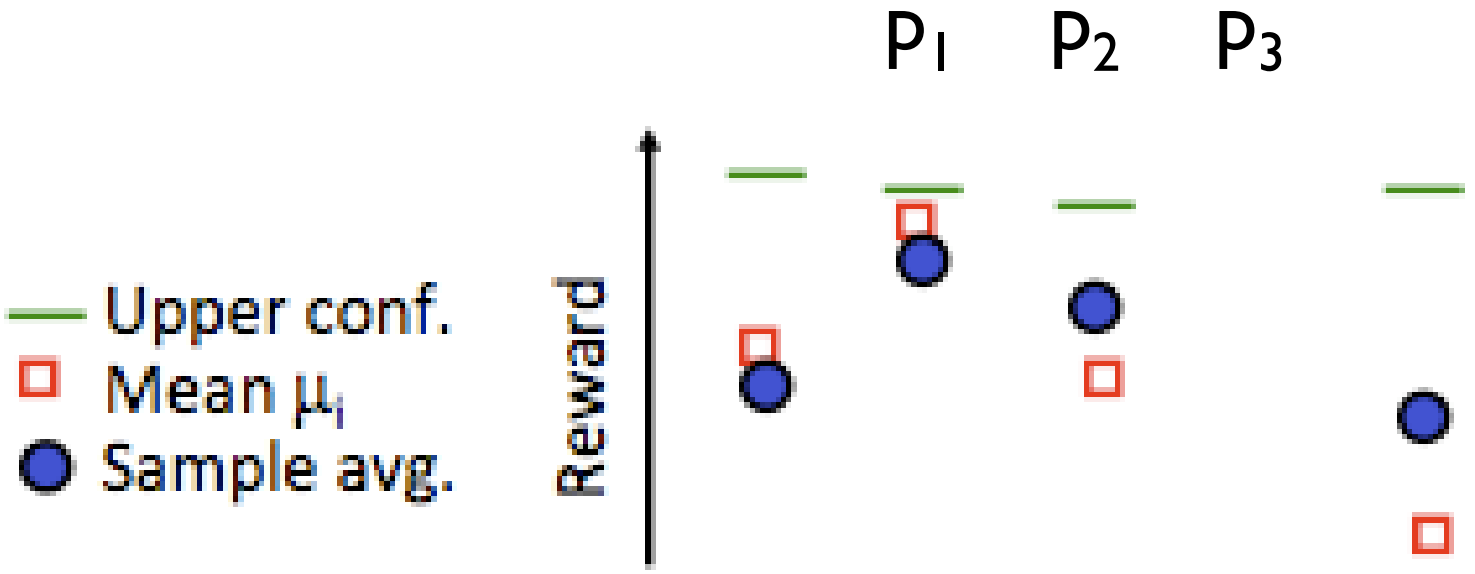
- One example: Modeling airflow over a plane wing
- A parameter vector is an arm
- Pulling an arm is costly
  - Difficult to actually calculate (computer models, PDEs…)
- Still want to maximize some kind of result across the arms

Developing an Algorithm

- Okay, so it's useful
- What kind of algorithm should we use?
- Random?
  - We've seen how well this works out
- Other obvious approaches are less applicable with infinitely many arms…

Developing an Algorithm

- We can reuse the ideas from the UCB_1 algorithm

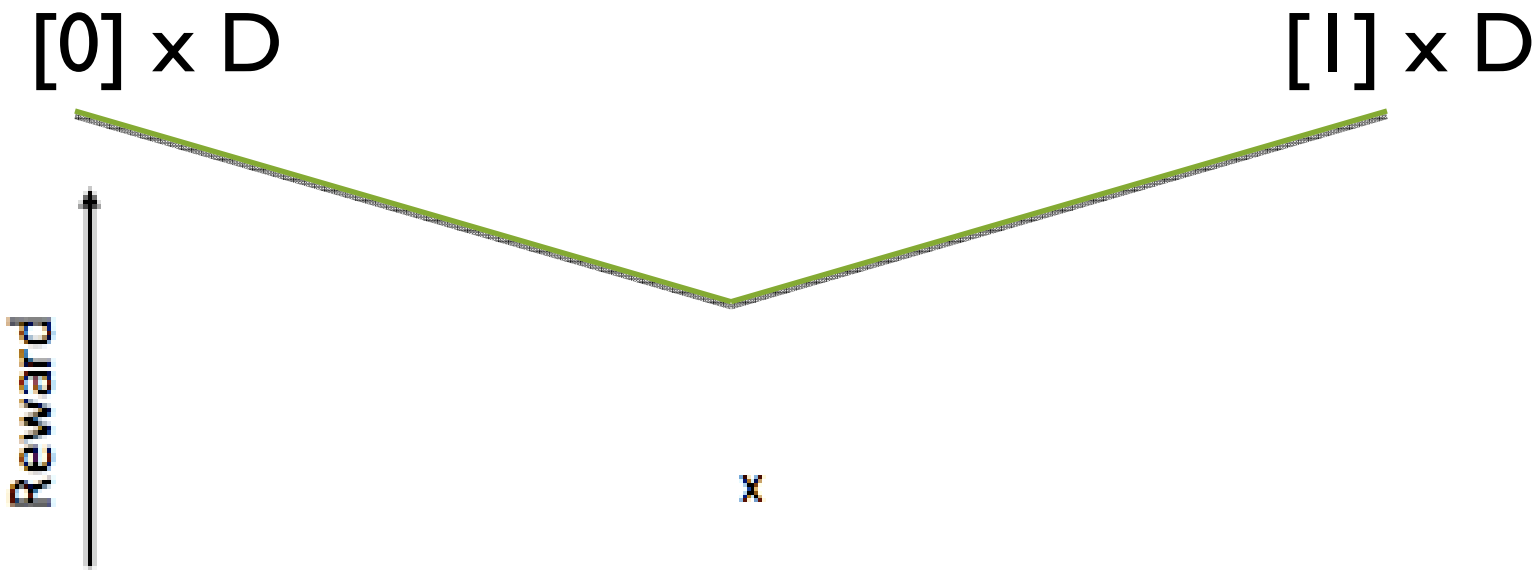


Adjustments Needed

- Not discrete arms, but a continuum
  - We will have need a UCB for all arms over the arm-space
- We can get some confidence about any pulled arm's neighbors because of Lipschitz

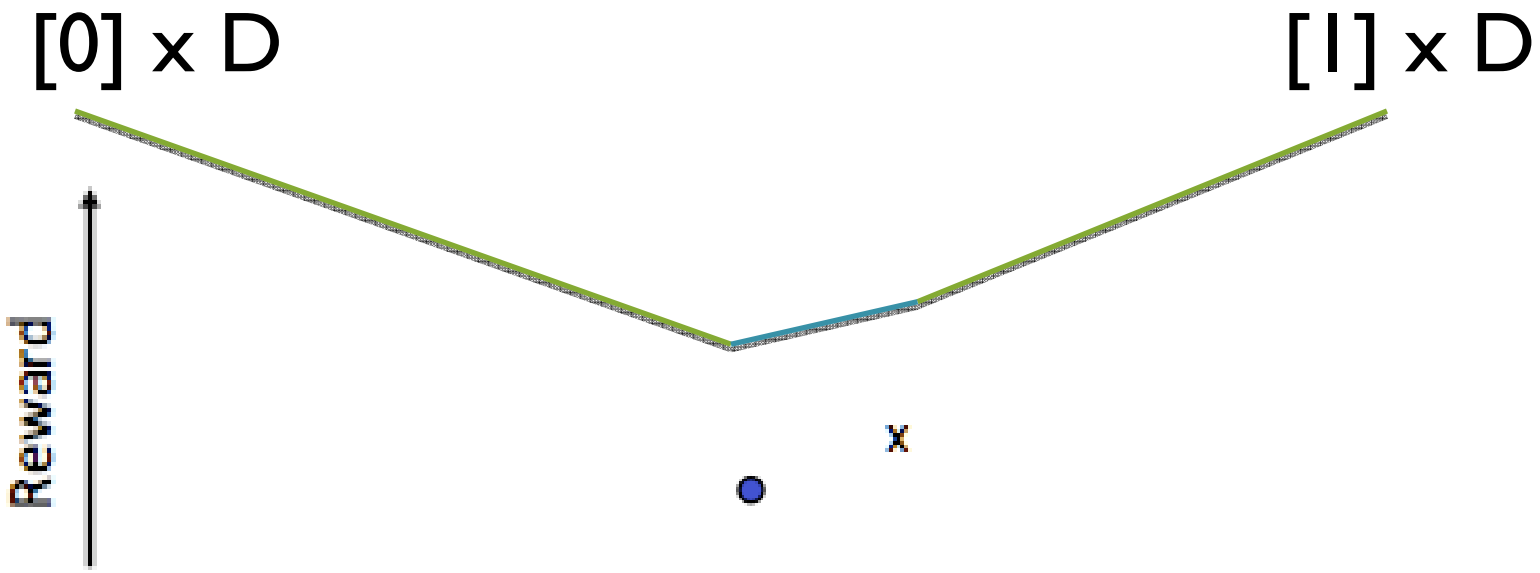
Stumbling Around

- Not discrete arms, but a continuum…



Stumbling Around

- New points affect their neighbors



Adjustments Needed

- We can also sharpen our estimates from nearby measurements
- Retain "optimism in the face of the unknown"
- General idea gotten…but how do we actually do it?

The Algorithm!

- Split the arm-space into regions
- Every time you pick an arm from a region, divide into more precise regions
- Keep track of how good every region is through results of itself and its children.

Setup for the Algorithm

- To remember regions, use a "Tree of Coverings"
- A node in the tree with height *h* and row-index *i* is represented as $P_{h,i}$ or just (h,i)
  - The children of $P_{h,i}$ are $P_{h+1,2i-1}$ and $P_{h+1,2i}$
  - The whole arm-space $X = P_{0,1}$
- The children of a node cover their parent

Setup for the Algorithm

- We always choose a leaf node, then add its children to the tree.
- Each node has a "score" – we pick a new leaf by going down the tree, going to the side with the greater score.
- Score:

$$B_{h,i}(n) = \min\{U_{h,i}(n), \max_{children}[B_{child}]\}$$

where $U_{h,i}(n)$ is the upper confidence bound for the tree node (h,i)

Setup for the Algorithm

- One more caveat – For any node (*h*,*i*), the diameter (determined by *d*, the dissimilarity function) of the smallest circle that bounds the node is less than $v_1 \rho^h$ for some parameters v,ρ
- A little more formally,

$$U_{h,i}(n) = \mu_{h,i}(n) + \text{Chernoff} + v_1 \rho^h$$

$$(\text{Chernoff} = \text{sqrt}[(2 \ln n) / N_{h,i}(n)]$$

)

Setup for the Algorithm

- Score:

$$B_{h,i}(n) = \min\{U_{h,i}(n), \max_{children}[B_{child}]\}$$

- What if you have no children?

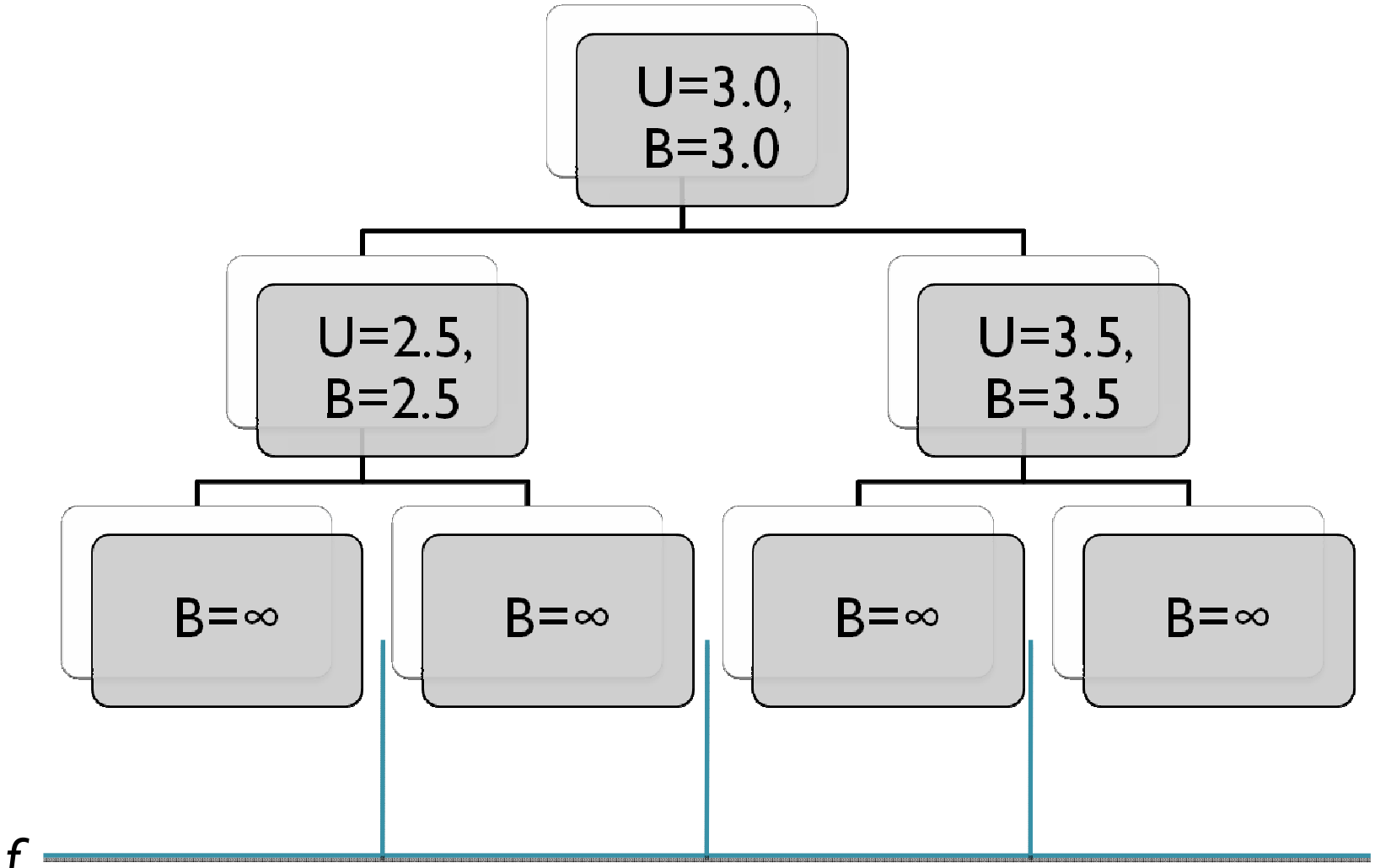
Setup for the Algorithm

- Score:

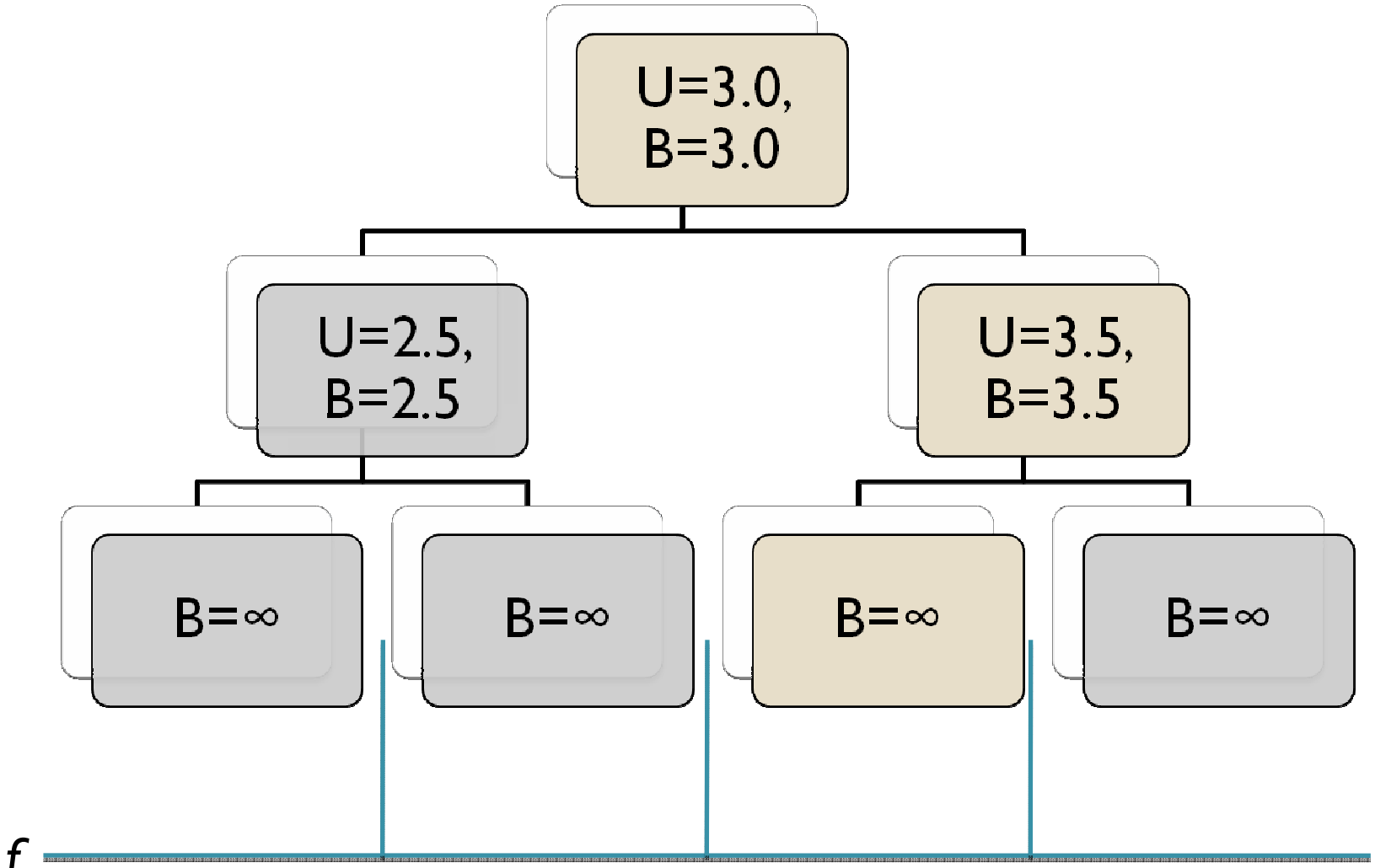
$$B_{h,i}(n) = \min\{U_{h,i}(n), \max_{children}[B_{child}]\}$$

- What if you haven't been picked yet?
- Optimism in the face of uncertainty!
  - Set B to infinity

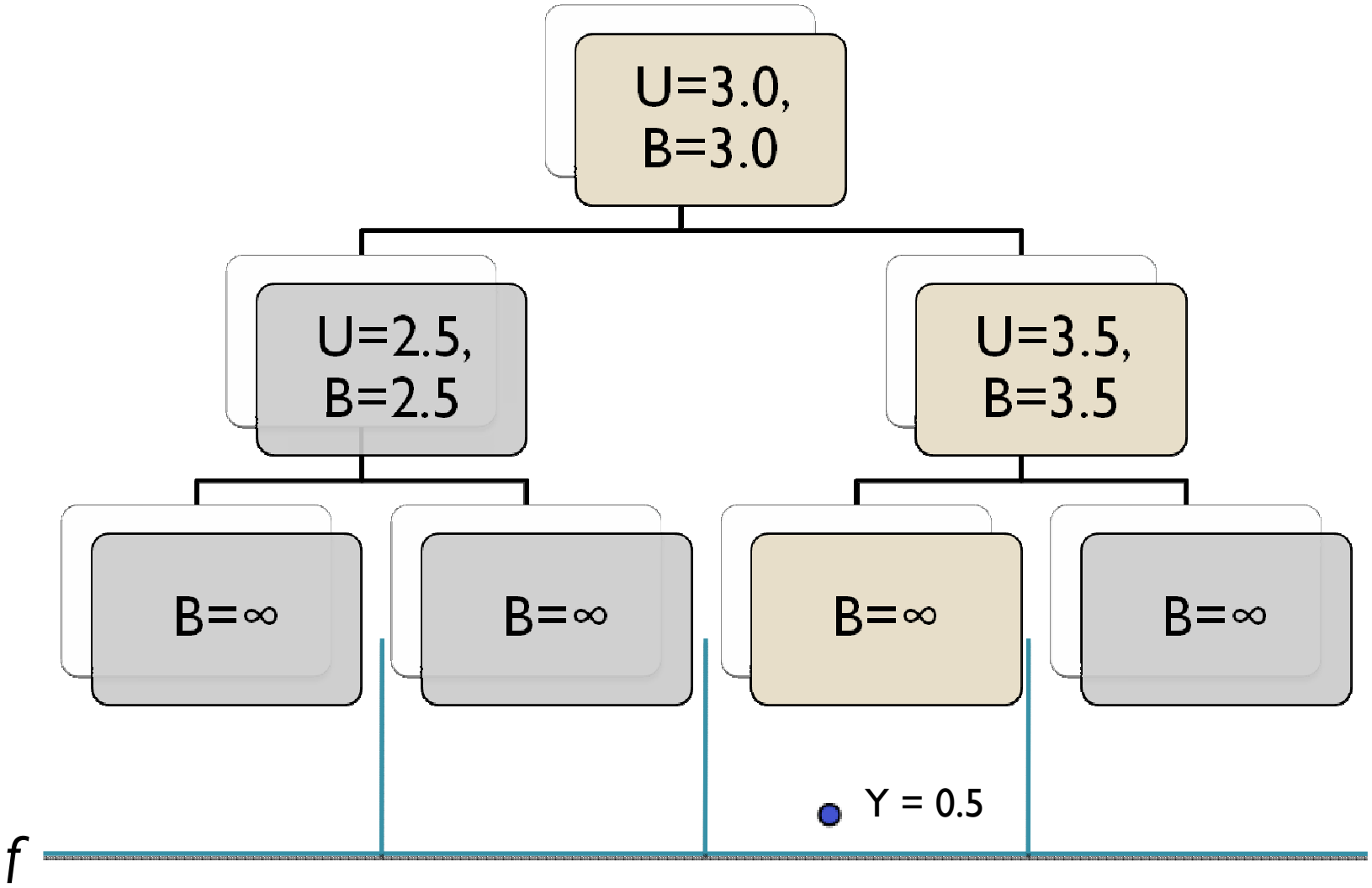
Algorithm Example



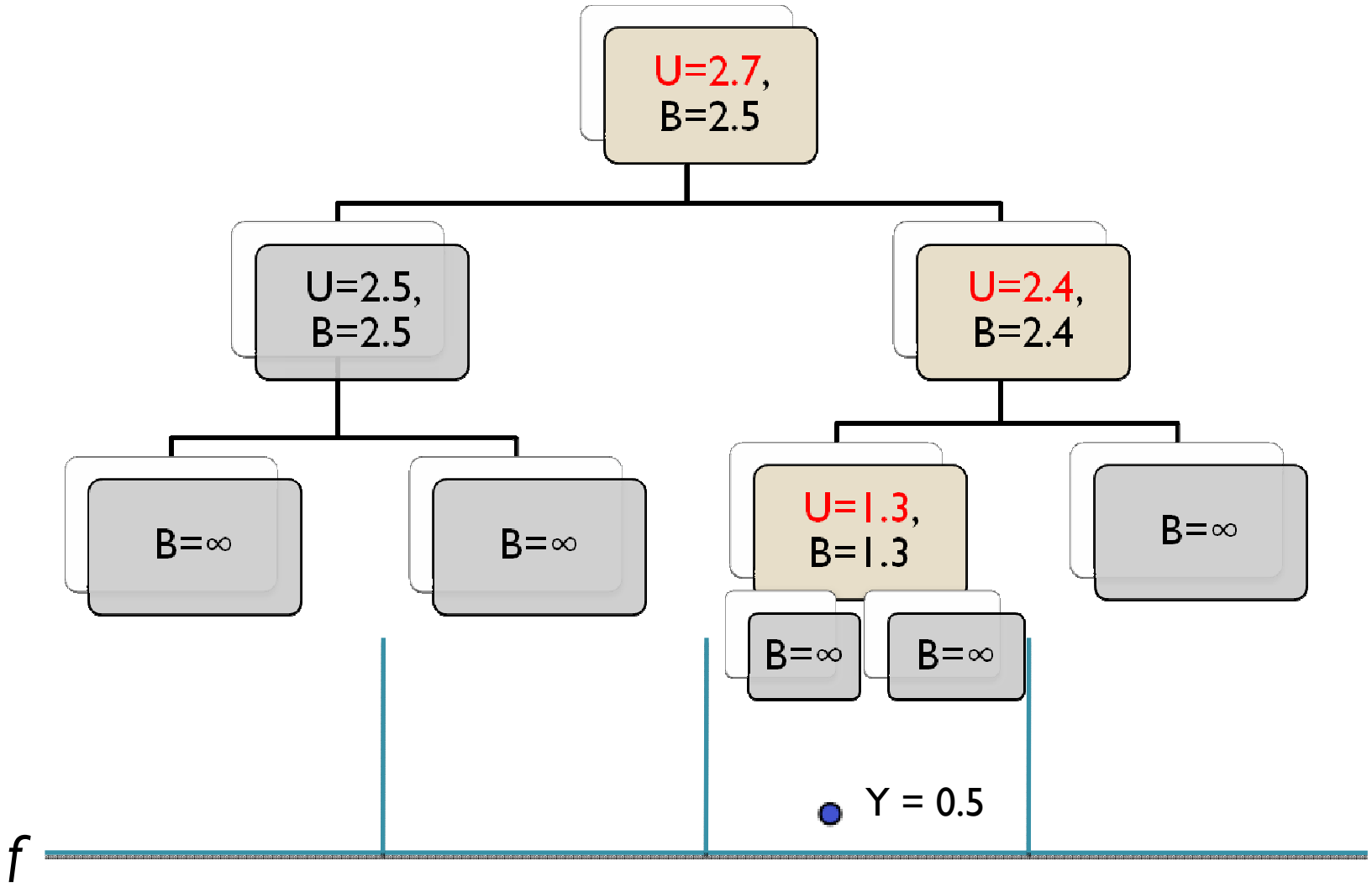
Algorithm Example



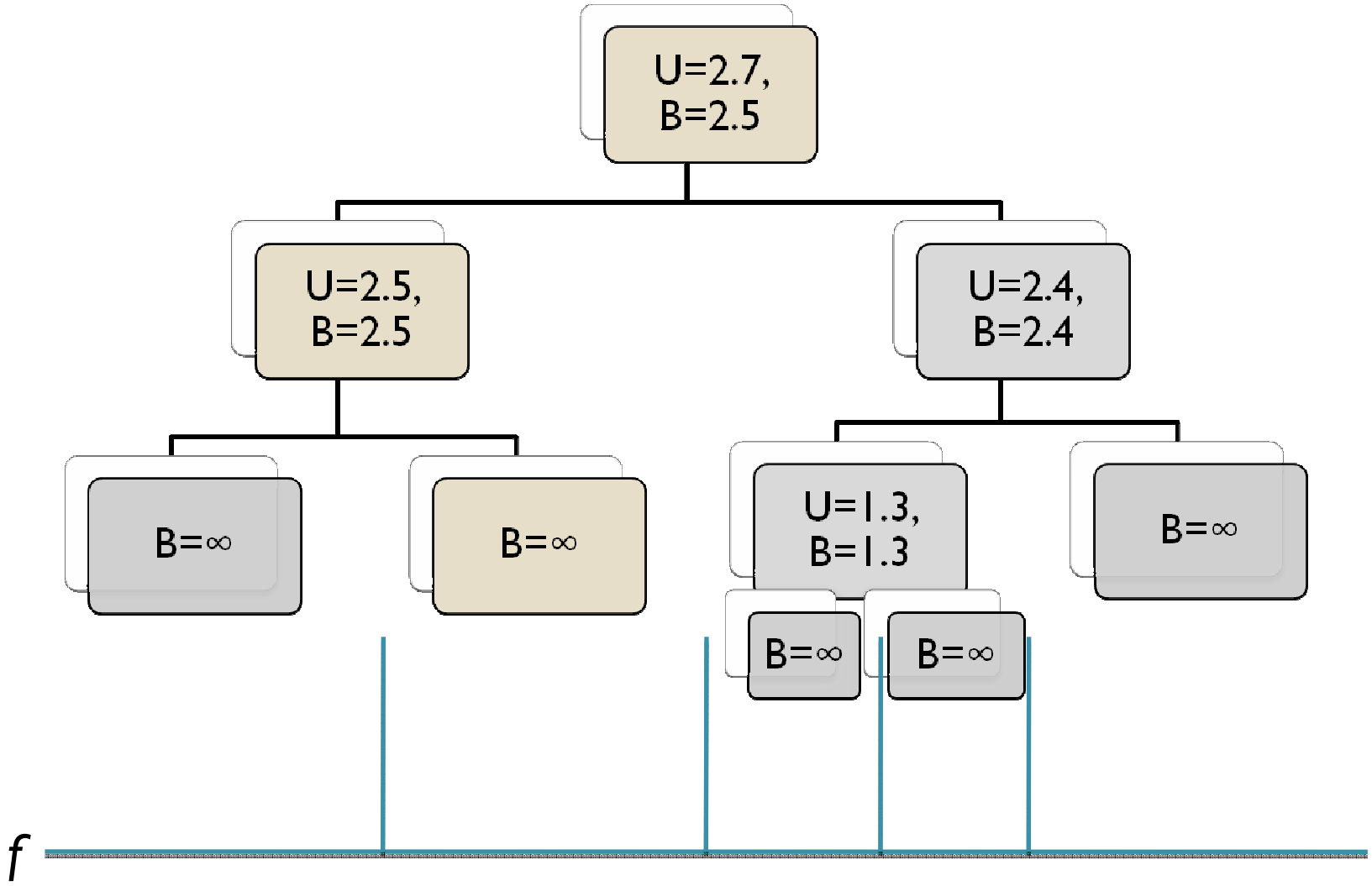
Algorithm Example



Algorithm Example



Algorithm Example



Observations

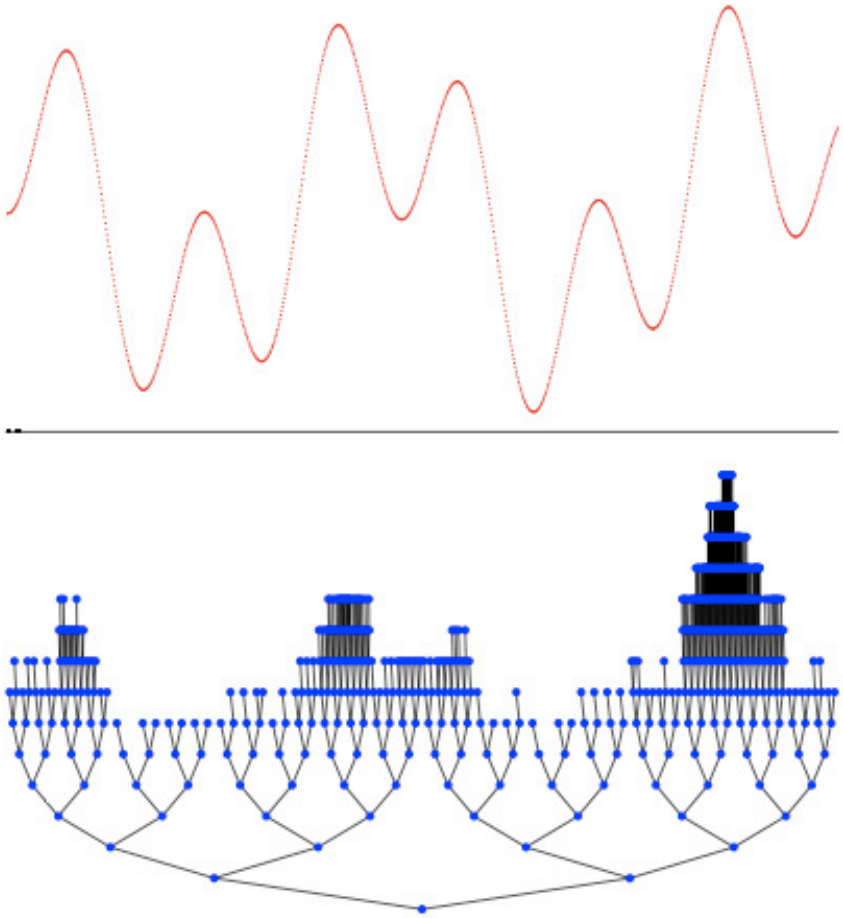
- Exploration comes from the pessimism of the B-score and the optimism of the unknown
- Exploitation comes from the optimism of the B-score and fast elimination of bad parts of the function

Numerical Results

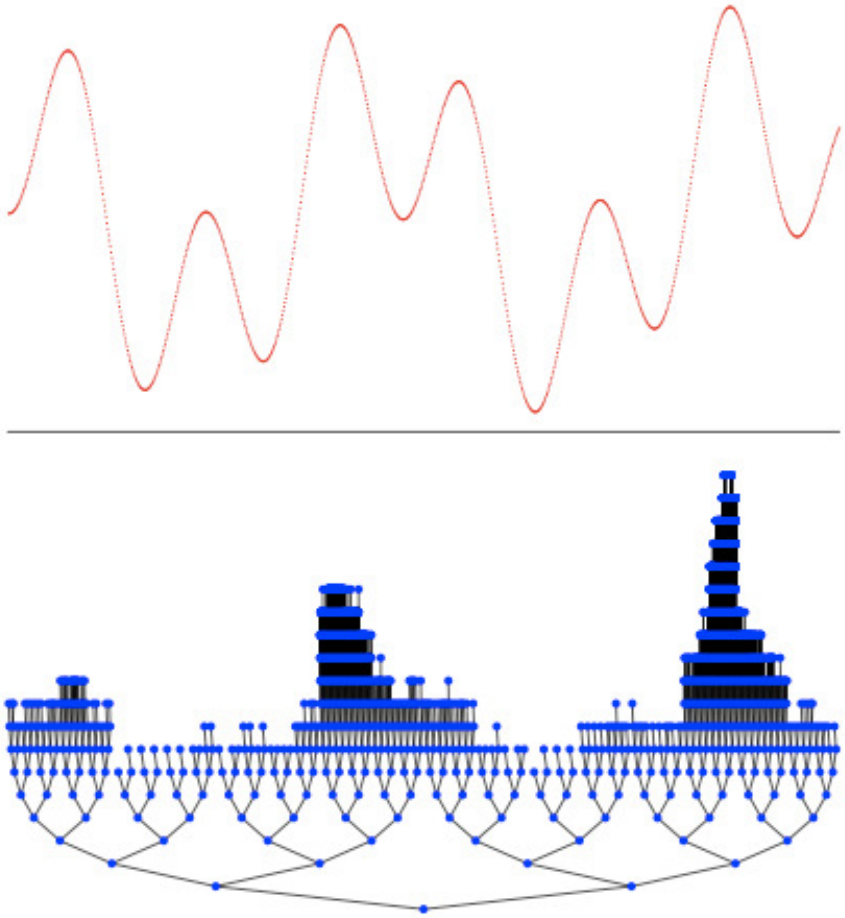
- The following is taken from another talk by the author, Sébastien Bubeck

Numerical Results

$n = 1000$



$n = 10000$



Regret Analysis

- Not going to go through all the math
  - If want, read the paper...
- Pretty similar to regret analysis of UCB_1
  - Number of times a bad arm is chosen is proportional to log(n) and inverse to difference to best arm
  - Add a lot of mess from the Lipschitzness
  - Actually, we only require "weak-Lipschitz", which is a sort of one-sided Lipschitz near the best arms

Regret Analysis

- Main result:
- $E(R_n) \leq C(d')\, n^{(d'+1)/(d'+2)} (\ln n)^{1/(d'+2)}$
  - C is some constant
  - d' is any number greater than d, and in most cases, can be equal to d

Regret Analysis

- $E(R_n) \leq C(d')\, n^{(d'+1)/(d'+2)} (\ln n)^{1/(d'+2)}$
- For high d, we get closer and closer to linear...
  - "The Curse of Dimensionality"
- This is proven to be tight! Tight!

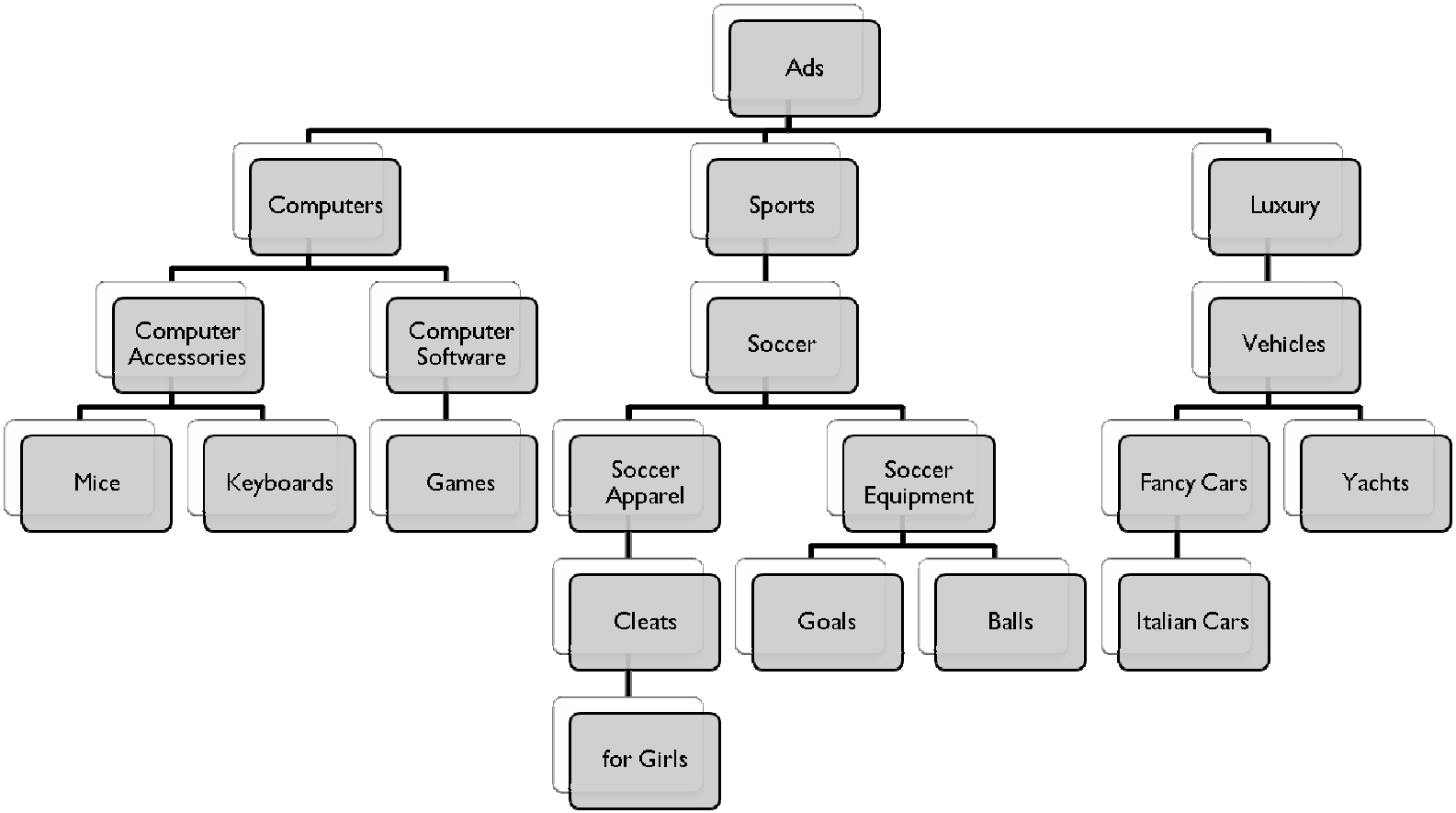
Dissimilarity Functions

- We've just been using straight distance
- *d* can be any metric
  - $d(x,y) = 0$ iff $x = y$
  - *d* must be symmetric
  - Triangle inequality
- With creative dissimilarity functions, this is surprisingly powerful!

Powerful Dissimilarities

- Suppose we go back to the example of online ads
- Ads sell all sorts of products (not quite infinite, but still more than we'd want to try individually!
- Can't we get information from knowing that some ads are related?

Online Product Sales



Online Product Sales

- Dissimilarity function should measure how, well, dissimilar two ads are.
- Can take the tree, weight the edges as, say, $1/h$, and compute distance
- Can now use the hierarchical algorithm!
- New dissimilarity functions add a lot of mileage...