Solution Set 6

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If you have not yet turned in the Problem Set, you should not consult these solutions.

1. As suggested, consider the directed Hamilton path problem on graph $G = (V, E)$. We are given G and two vertices s and t. The universe will be the edges E. Let $\mathcal{I}_1$ be the matroid consisting of those sets with at most one outgoing edge from each node (except for t). The only thing that is not trivial that needs to be checked is the exchange property. Clearly if we have two independent sets A and B with $|A| > |B|$, then A must contain at least one edge incident to a vertex that is not touched by any edge in B. This edge can be added to B resulting in an independent set. The second matroid $\mathcal{I}_2$ consists of sets with at most one incoming edge to each node (except for s). This is a matroid for the same reason $\mathcal{I}_1$ is. And let $\mathcal{I}_3$ be the graphic matroid for graph G. For each of the three matroids it is easy to decide in polynomial time whether a given set is an independent set.

   We claim that any independent set of size $|V|-1$ in the intersection of all three matroids must be a Hamilton path from s to t. It must be a spanning tree (because it is an independent set of cardinality $|V| - 1$ in the graphic matroid), and every vertex can have at most one incoming and at most one outgoing edge. Thus it is a directed path, and it must start at s and end at t because s is the only vertex that is not required to have an incoming edge in $\mathcal{I}_2$ and t is the only vertex not required to have an outgoing edge in $\mathcal{I}_1$, and the cardinality required that every vertex other than those exceptions have an outgoing and incoming edge.

2. Consider repeating the two operations until they cannot be applied. For operations 1 we observe that any vertex of degree greater than k must be in the vertex cover, since it is not possible to cover all the incident edges with the other endpoints (there are at least $k + 1$ of them). So we can safely remove such a high-degree vertex and reduce the parameter k by one. Also, an isolated vertex can simply be removed as it will not be in any (minimal) vertex cover. When this process stops, with final parameter $k' \leq k$, if the graph has more than k^2 edges, then we know that there cannot be a vertex cover of size k, since each vertex has degree at most k so it can cover at most k edges (and thus k vertices in any cover can cover at most k^2 edges). If so, we simply output a graph with no vertex cover of size k' – e.g. a single edge, and $k' = 0$.

   Otherwise, there are two endpoints to each edge (and no isolated vertices), so there are at most $2k^2$ vertices when we end, and there is a vertex cover of size k' in this graph iff there was a vertex cover of size k in the original graph.

3. (a) Our linear program has objective function $\sum_{p \in P} x_p$ which we want to maximize, and constraints
      - $\sum_{p \ni e} x_p \leq 1$ for all edges e, and

- $x_p \geq 0$

(b) The dual linear program has objective function $\sum_e y_e$ which we want to minimize, and constraints

- $\sum_{e \in p} y_e \geq 1$ for all $p \in P$, and
- $x_p \geq 0$

This LP asks us to assign edge weights so that every path in P has length at least 1 (so the shortest path has length 1), while minimizing the total weight.

(c) Given a purported feasible solution to the above LP (the dual), which has exponentially many constraints, we can find a violated constraint (if there is one) as follows: first, see if any y_e is negative. If so, that is a violated constraint. Next we compute shortest s-t paths using Dijkstra's algorithm, and the edge weights of the purported feasible solution. If the shortest path length is less than 1, then we take that path, and report that the associated constraint is violated. Otherwise, all the constraints are satisfied.