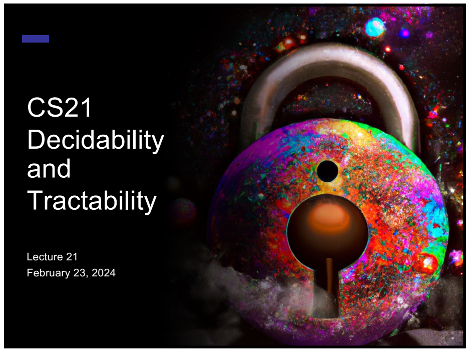


1

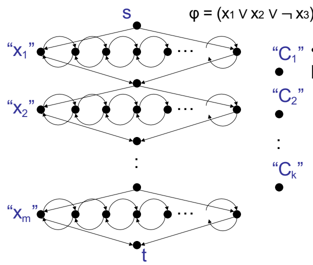
Outline

- NP-complete problems: independent set, vertex cover, clique…
- NP-complete problems: Hamilton path and cycle, Traveling Salesperson Problem
- NP-complete problems: Subset Sum
- NP-complete problems: NAE-3-SAT, max cut

2

HAMPATH is NP-complete

$\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \ldots$



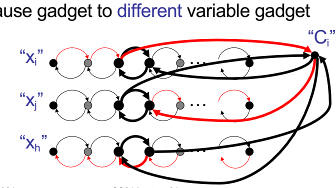
- NO maps to NO?
  - try to translate path into satisfying assignment
  - if path has "intended" form, OK.

3

HAMPATH is NP-complete

- What can go wrong?
  - path has "intended form" unless return from clause gadget to different variable gadget

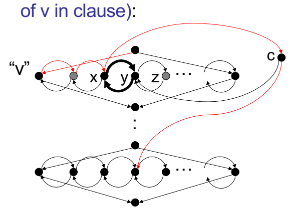
we will argue that this cannot happen:



4

HAMPATH is NP-complete

Case 1 (positive occurrence of v in clause):

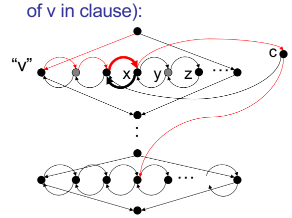


- path must visit y
- must enter from x, z, or c
- must exit to z (x is taken)
- x, c are taken. can't happen

5

HAMPATH is NP-complete

Case 2 (negative occurrence of v in clause):



- path must visit y
- must enter from x or z
- must exit to z (x is taken)
- x is taken. can't happen

6

Undirected Hamilton Path

- HAMPATH refers to a directed graph.
- Is it easier on an undirected graph?
- A language (decision problem):

  UHAMPATH = {(G, s, t) : undirected G has a Hamilton path from s to t}

7

UHAMPATH is NP-complete

Theorem: the following language is NP-complete:

UHAMPATH = {(G, s, t) : undirected graph G has a Hamilton path from s to t}

- Proof:
  – Part 1: UHAMPATH $\in$ NP. Proof?
  – Part 2: UHAMPATH is NP-hard.
    - reduce from?

8

UHAMPATH is NP-complete

- We are reducing from the language:

  HAMPATH = {(G, s, t) : directed graph G has a Hamilton path from s to t}

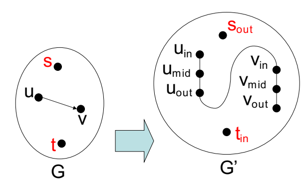
  to the language:

  UHAMPATH = {(G, s, t) : undirected graph G has a Hamilton path from s to t}

9

UHAMPATH is NP-complete

- The reduction:



- replace each node with three (except s, t)
- (u_{in}, u_{mid})
- (u_{mid}, u_{out})
- (u_{out}, v_{in}) iff G has (u,v)

10

UHAMPATH is NP-complete

- Does the reduction run in poly-time?
- YES maps to YES?
  – Hamilton path in G: $s, u_1, u_2, u_3, \ldots, u_k, t$
  – Hamilton path in G':

  $s_{out}, (u_1)_{in}, (u_1)_{mid}, (u_1)_{out}, (u_2)_{in}, (u_2)_{mid}, (u_2)_{out}, \ldots (u_k)_{in}, (u_k)_{mid}, (u_k)_{out}, t_{in}$

11

UHAMPATH is NP-complete

- NO maps to NO?
  – Hamilton path in G':

  $s_{out}, v_1, v_2, v_3, v_4, v_5, v_6, \ldots, v_{k-2}, v_{k-1}, v_k, t_{in}$

  – $v_1 = (u_{i1})_{in}$ for some i_1 (only edges to ins)
  – $v_2 = (u_{i1})_{mid}$ for some i_1 (only way to enter mid)
  – $v_3 = (u_{i1})_{out}$ for some i_1 (only way to exit mid)
  – $v_4 = (u_{i2})_{in}$ for some i_2 (only edges to ins)
  – …
  – Hamilton path in G: $s, u_{i1}, u_{i2}, u_{i3}, \ldots, u_{ik}, t$

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Undirected Hamilton Cycle

- Definition: given a undirected graph G = (V, E), a Hamilton cycle in G is a cycle in G that touches every node exactly once.
- Is finding one easier than finding a Hamilton path?
- A language (decision problem):
  UHAMCYCLE = {G : G has a Hamilton cycle}

UHAMCYCLE is NP-complete

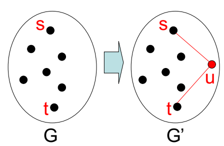
Theorem: the following language is NP-complete:

UHAMCYCLE = {G: G has a Hamilton cycle}

- Proof:
  - Part 1: UHAMCYCLE ∈ NP. Proof?
  - Part 2: UHAMCYCLE is NP-hard.
    - reduce from?

UHAMCYCLE is NP-complete

- The reduction (from UHAMPATH):



- H. path from s to t implies H. cycle in G'
- H. cycle in G' must visit u via red edges
- removing red edges gives H. path from s to t in G

Traveling Salesperson Problem

- Definition: given n cities $v_1, v_2, \ldots, v_n$ and inter-city distances $d_{i,j}$ a TSP tour in G is a permutation π of {1…n}. The tour's length is $\Sigma_{i = 1\ldots n}\, d_{\pi(i),\pi(i+1)}$ (where n+1 means 1).
- A search problem:
  given the $\{d_{i,j}\}$, find the shortest TSP tour
- corresponding language (decision problem):
  TSP = {($\{d_{i,j} : 1 \le i<j \le n\}$, k) : these cities have a TSP tour of length ≤ k}

TSP is NP-complete

Theorem: the following language is NP-complete:

TSP = {($\{d_{i,j} : 1 \le i<j \le n\}$, k) : these cities have a TSP tour of length ≤ k}

- Proof:
  - Part 1: TSP ∈ NP. Proof?
  - Part 2: TSP is NP-hard.
    - reduce from?

TSP is NP-complete

- We are reducing from the language:

UHAMCYCLE = {G : G has a Hamilton cycle}

to the language:

TSP = {($\{d_{i,j} : 1 \le i<j \le n\}$, k) : these cities have a TSP tour of length ≤ k}

TSP is NP-complete

- The reduction:
  - given G = (V, E) with n nodes

  produce:
  - n cities corresponding to the n nodes
  - $d_{u,v}$ = 1 if (u, v) ∈ E
  - $d_{u,v}$ = 2 if (u, v) ∉ E
  - set k = n

TSP is NP-complete

- YES maps to YES?
  - if G has a Hamilton cycle, then visiting cities in that order gives TSP tour of length n
- NO maps to NO?
  - if TSP tour of length ≤ n, it must have length exactly n.
  - all distances in tour are 1. Must be edges between every successive pair of cities in tour.

Hamilton Path

- Definition: given a directed graph G = (V, E), a Hamilton path in G is a directed path that touches every node exactly once.

- A language (decision problem):

  HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

HAMPATH is NP-complete

Theorem: the following language is NP-complete:

HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

- Proof:
  - Part 1: HAMPATH ∈ NP. Proof?
  - Part 2: HAMPATH is NP-hard.
    - reduce from?

HAMPATH is NP-complete

- We are reducing from the language:

  3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

  to the language:

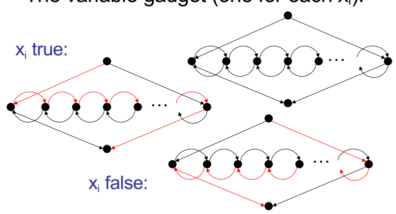
  HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

HAMPATH is NP-complete

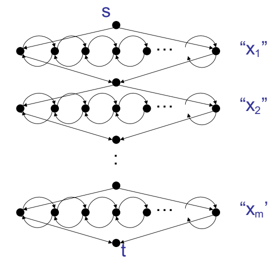
- We want to construct a graph from φ with the following properties:
  - a satisfying assignment to φ translates into a Hamilton Path from s to t
  - a Hamilton Path from s to t can be translated into a satisfying assignment for φ
- We will build the graph up from pieces called gadgets that “simulate” the clauses and variables of φ.

HAMPATH is NP-complete

- The variable gadget (one for each x_i):



HAMPATH is NP-complete



- path from s to t translates into a truth assignment to $x_1 \ldots x_m$
- why must the path be of this form?

HAMPATH is NP-complete

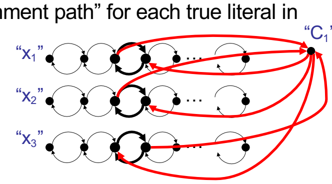
$\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$

- How to ensure that all k clauses are satisfied?
- need to add nodes
  - can be visited in path if the clause is satisfied
  - if visited in path, implies clause is satisfied by the assignment given by path through variable gadgets

HAMPATH is NP-complete

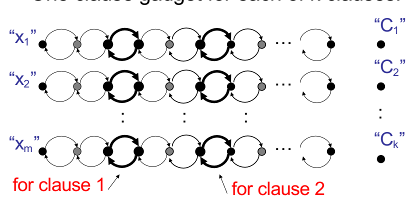
$\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$

- Clause gadget allows "detour" from "assignment path" for each true literal in clause



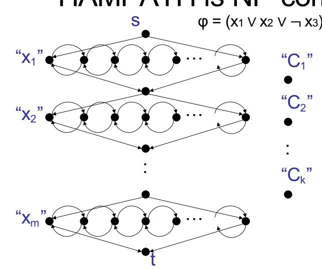
HAMPATH is NP-complete

- One clause gadget for each of k clauses:



HAMPATH is NP-complete

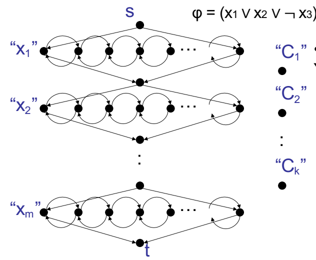
$\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \ldots$



- $f(\varphi)$ is this graph (edges to/from clause nodes not pictured)
- f poly-time computable?
  - # nodes = $O(km)$

HAMPATH is NP-complete

$\varphi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_4 \lor x_3) \ldots$



- YES maps to YES?
  - first form path from satisfying assign.
  - pick true literal in each clause and add detour

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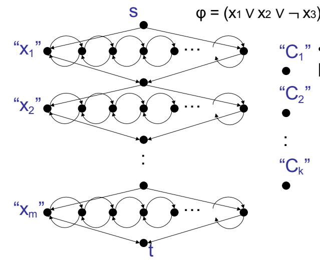
HAMPATH is NP-complete

$\varphi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_4 \lor x_3) \ldots$



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32

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- What can go wrong?
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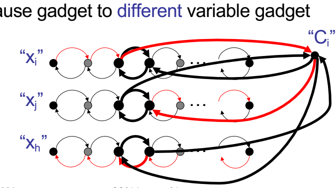
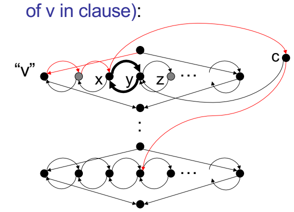
we will argue that this cannot happen:



33

HAMPATH is NP-complete

Case 1 (positive occurrence of v in clause):



- path must visit y
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- must exit to z (x is taken)
- x, c are taken. can't happen

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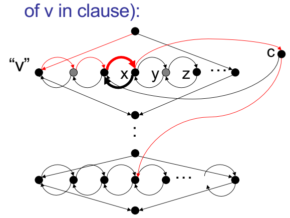
HAMPATH is NP-complete

Case 2 (negative occurrence of v in clause):



- path must visit y
- must enter from x or z
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- x is taken. can't happen

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- Is it easier on an undirected graph?
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36

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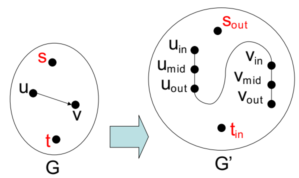
UHAMPATH is NP-complete

- The reduction:



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UHAMPATH is NP-complete

- NO maps to NO?
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  - $v_3 = (u_{i1})_{out}$ for some i_1 (only way to exit mid)
  - $v_4 = (u_{i2})_{in}$ for some i_2 (only edges to ins)
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  - Hamilton path in G: $s, u_{i1}, u_{i2}, u_{i3}, \ldots, u_{ik}, t$

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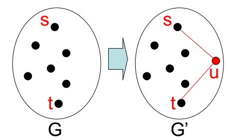
- The reduction (from UHAMPATH):



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- A search problem: given the $\{d_{i,j}\}$, find the shortest TSP tour
- corresponding language (decision problem): TSP = $\{(\{d_{i,j} : 1 \le i<j \le n\}, k)$: these cities have a TSP tour of length $\le k\}$

TSP is NP-complete

Theorem: the following language is NP-complete:

TSP = $\{(\{d_{i,j} : 1 \le i<j \le n\}, k)$: these cities have a TSP tour of length $\le k\}$

- Proof:
  - Part 1: TSP ∈ NP. Proof?
  - Part 2: TSP is NP-hard.
    - reduce from?

TSP is NP-complete

- We are reducing from the language:

UHAMCYCLE = {G : G has a Hamilton cycle}

to the language:

TSP = $\{(\{d_{i,j} : 1 \le i<j \le n\}, k)$: these cities have a TSP tour of length $\le k\}$

TSP is NP-complete

- The reduction:
  - given G = (V, E) with n nodes

  produce:
  - n cities corresponding to the n nodes
  - $d_{u,v} = 1$ if $(u, v) \in E$
  - $d_{u,v} = 2$ if $(u, v) \notin E$
  - set k = n

TSP is NP-complete

- YES maps to YES?
  - if G has a Hamilton cycle, then visiting cities in that order gives TSP tour of length n
- NO maps to NO?
  - if TSP tour of length ≤ n, it must have length exactly n.
  - all distances in tour are 1. Must be edges between every successive pair of cities in tour.

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Subset Sum

- A language (decision problem):

  SUBSET-SUM = {(S = {a_1, a_2, a_3, …, a_k}, B) : there is a subset of S that sums to B}

- example:
  - S = {1, 7, 28, 3, 2, 5, 9, 32, 41, 11, 8}
  - B = 30
  - 30 = 7 + 3 + 9 + 11. yes.

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Subset Sum

SUBSET-SUM = {(S = {a_1, a_2, a_3, …, a_k}, B) : there is a subset of S that sums to B}

- Is this problem NP-complete? in P?
- Problem set: in TIME(B·poly(k))

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SUBSET-SUM is NP-complete

Theorem: the following language is NP-complete:

SUBSET-SUM = {(S = {a_1, a_2, a_3, …, a_k}, B) : there is a subset of S that sums to B}

our reduction had better produce super-polynomially large B (unless we want to prove P=NP)

- Proof:
  - Part 1: SUBSET-SUM ∈ NP.
  - Part 2: SUBSET-SUM is NP-hard.
    - reduce from?

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SUBSET-SUM is NP-complete

- We are reducing from the language:

  3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

  to the language:

  SUBSET-SUM = {(S = {a_1, a_2, a_3, …, a_k}, B) : there is a subset of S that sums to B}

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SUBSET-SUM is NP-complete

- $\varphi = (x_1 \lor x_2 \lor \neg x_3)\land(\neg x_1 \lor x_4 \lor x_3)\land \ldots \land(\ldots)$
- Need integers to play the role of truth assignments
- For each variable x_i include two integers in our set S:
  - x_i^{TRUE} and x_i^{FALSE}
- set B so that exactly one must be in sum

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SUBSET-SUM is NP-complete

x_1^{TRUE} = 1 0 0 0 … 0
x_1^{FALSE} = 1 0 0 0 … 0
x_2^{TRUE} = 0 1 0 0 … 0
x_2^{FALSE} = 0 1 0 0 … 0
…
x_m^{TRUE} = 0 0 0 0 … 1
x_m^{FALSE} = 0 0 0 0 … 1
B = 1 1 1 1 … 1

- every choice of one from each $(x_i^{TRUE}, x_i^{FALSE})$ pair sums to B
- every subset that sums to B must choose one from each $(x_i^{TRUE}, x_i^{FALSE})$ pair

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SUBSET-SUM is NP-complete

- $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$
- Need to force subset to "choose" at least one true literal from each clause
- Idea:
  - add more digits
  - one digit for each clause
  - set B to force each clause to be satisfied.

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SUBSET-SUM is NP-complete

- $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge \ldots \wedge (\ldots)$

x_1^{TRUE} = 1 0 0 0 … 0 1
x_1^{FALSE} = 1 0 0 0 … 0 0
x_2^{TRUE} = 0 1 0 0 … 0 1
x_2^{FALSE} = 0 1 0 0 … 0 0
x_3^{TRUE} = 0 0 1 0 … 0 0
x_3^{FALSE} = 0 0 1 0 … 0 1
…
B = 1 1 1 1 … 1 ? ? ? ?

clause 1
clause 2
clause 3
:
clause k

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SUBSET-SUM is NP-complete

- B = 1 1 1 1 … 1 ? ? ? ?
- if clause i is satisfied sum might be 1, 2, or 3 in corresponding column.
- want ? to "mean" ≥ 1
- solution: set ? = 3
- add two "filler" elements for each clause i:
- $FILL1_i$ = 0 0 0 0 … 0 0 … 0 1 0 … 0
- $FILL2_i$ = 0 0 0 0 … 0 0 … 0 1 0 … 0

column for clause i

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SUBSET-SUM is NP-complete

- Reduction: m variables, k clauses
  - for each variable x_i:
    - x_i^{TRUE} has ones in positions k + i and {j : clause j includes literal x_i}
    - x_i^{FALSE} has ones in positions k + i and {j : clause j includes literal $\neg x_i$}
  - for each clause i:
    - $FILL1_i$ and $FILL2_i$ have one in position i
  - bound B has 3 in positions 1…k and 1 in positions k+1…k+m

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