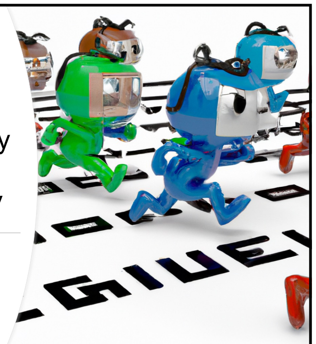
CS21 Decidability and Tractability



Lecture 16
February 9, 2024

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Complexity

- Complexity Theory = study of what is computationally feasible (or tractable) with limited resources:
  - running *time* — main focus
  - storage *space*
  - number of *random bits*
  - degree of *parallelism*
  - rounds of *interaction*
  - *others…*

  (number of random bits through others: not in this course)

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Worst-case analysis

- Always measure resource (e.g. running time) in the following way:
  - as a function of the input length
  - value of the fn. is the maximum quantity of resource used over all inputs of given length
  - called "worst-case analysis"
- "input length" is the length of input string, which might encode another object with a separate notion of size

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Time complexity

Definition: the running time ("time complexity") of a TM M is a function

$$f:\mathbf{N} \rightarrow \mathbf{N}$$

where f(n) is the maximum number of steps M uses on any input of length n.

- "M runs in time f(n)," "M is a f(n) time TM"

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Time complexity

- Example: TM M deciding $L = \{0^k1^k : k \geq 0\}$.

On input x:

- scan tape left-to-right, reject if 0 to right of 1 — # steps?
- repeat while 0's, 1's on tape:
  - scan, crossing off one 0, one 1 — # steps?
- if only 0's or only 1's remain, reject; if neither 0's nor 1's remain, accept — # steps?

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Time complexity

- We do not care about fine distinctions
  - e.g. how many additional steps M takes to check that it is at the left of tape
- We care about the behavior on large inputs
  - general-purpose algorithm should be "scalable"
  - overhead for e.g. initialization shouldn't matter in big picture

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Time complexity

- Measure time complexity using asymptotic notation ("big-oh notation")
  - disregard lower-order terms in running time
  - disregard coefficient on highest order term
- example:

$$f(n) = 6n^3 + 2n^2 + 100n + 102781$$

  - "f(n) is order n^3"
  - write $f(n) = O(n^3)$

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Asymptotic notation

Definition: given functions $f,g:\mathbf{N} \rightarrow \mathbf{R}^+$, we say $f(n) = O(g(n))$ if there exist positive integers c, n_0 such that for all $n \geq n_0$

$$f(n) \leq cg(n).$$

- meaning: f(n) is (asymptotically) less than or equal to g(n)
- if g > 0 can assume $n_0 = 0$, by setting

$$c' = \max_{0 \leq n \leq n_0}\{c, f(n)/g(n)\}$$

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Asymptotic notation facts

each bound asymptotically less than next

- "logarithmic": $O(\log n)$
  - $\log_b n = (\log_2 n)/(\log_2 b)$
  - so $\log_b n = O(\log_2 n)$ for any constant b; therefore suppress base when write it
- "polynomial": $O(n^c) = n^{O(1)}$
  - also: $c^{O(\log n)} = O(n^{c'}) = n^{O(1)}$
- "exponential": $O(2^{n^\delta})$ for $\delta > 0$

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Time complexity

On input x:	
• scan tape left-to-right, reject if 0 to right of 1	O(n) steps
• repeat while 0's, 1's on tape:	≤ n repeats
• scan, crossing off one 0, one 1	O(n) steps
• if only 0's or only 1's remain, reject; if neither 0's nor 1's remain, accept	O(n) steps

- total = $O(n) + nO(n) + O(n) = O(n^2)$

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Time complexity

- Recall:
  - language is a set of strings
  - a complexity class is a set of languages
  - complexity classes we've seen:
    - Regular Languages, Context-Free Languages, Decidable Languages, RE Languages, co-RE languages

Definition: TIME(t(n)) = {L : there exists a TM M that decides L in time O(t(n))}

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Time complexity

- We saw that $L = \{0^k1^k : k \geq 0\}$ is in TIME(n^2).
- Book: it is also in TIME(n log n) by giving a more clever algorithm
- Can prove: There does not exist a (single tape) TM which decides L in time (asymptotically) less than n log n
- How about on a multitape TM?

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Time complexity

- 2-tape TM M deciding $L = \{0^k1^k : k \geq 0\}$.

On input x:	
• scan tape left-to-right, reject if 0 to right of 1	O(n)
• scan 0's on tape 1, copying them to tape 2	O(n)
• scan 1's on tape 1, crossing off 0's on tape 2	O(n)
• if all 0's crossed off before done with 1's reject	
• if 0's remain after done with ones, reject; otherwise accept.	total: 3*O(n) = O(n)

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Multitape TMs

- Convenient to "program" multitape TMs rather than single ones
  - equivalent when talking about decidability
  - not equivalent when talking about time complexity

Theorem: Let t(n) satisfy t(n) ≥ n. Every multi-tape TM running in time t(n) has an equivalent TM running in time $O(t(n)^2)$.

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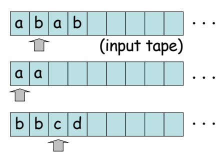
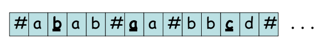
Multitape TMs

simulation of k-tape TM by single-tape TM:



- add new symbol **x** for each old x
- marks location of "virtual heads"



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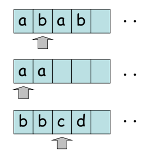
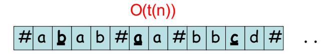
Multitape TMs



Repeat: O(t(n)) times

- scan tape, remembering the symbols under each virtual head in the state
  O(k t(n)) = O(t(n))
- make changes to reflect 1 step of M; if hit #, shift to right to make room.
  O(k t(n)) = O(t(n))

when M halts, erase all but 1st string
O(t(n))



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Multitape TMs

- Moral: feel free to use k-tape TMs, but be aware of slowdown in conversion to TM
  - note: if $t(n) = O(n^c)$ then $t(n)^2 = O(n^{2c}) = O(n^{c'})$
  - note: if $t(n) = O(2^{n^\delta})$ for $\delta > 0$ then $t(n)^2 = O(2^{2n^\delta}) = O(2^{n^{\delta'}})$ for $\delta' > 0$
- high-level operations you are used to using can be simulated by TM with only polynomial slowdown
  - e.g., copying, moving, incrementing/decrementing, arithmetic operations +, -, *, /

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Extended Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an efficient algorithm is:

> <u>The "extended" Church-Turing Thesis</u>
>
> everything we can compute in time t(n) on a physical computer can be computed on a Turing Machine in time $t(n)^{O(1)}$ (polynomial slowdown)

- quantum computers challenge this belief

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Time Complexity

- interested in a coarse classification of problems. For this purpose,
  - treat any polynomial running time as "efficient" or "tractable"
  - treat any exponential running time as inefficient or "intractable"

Key definition: "P" or "polynomial-time" is

$$P = \cup_{k \geq 1} \text{TIME}(n^k)$$

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Time Complexity

- Why polynomial-time?
  - insensitive to particular deterministic model of computation chosen
  - closed under modular composition
  - empirically: qualitative breakthrough to achieve polynomial running time is followed by quantitative improvements from impractical (e.g. n^{100}) to practical (e.g. n^3 or n^2)

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Examples of languages in P

- Recall: positive integers x, y are relatively prime if their Greatest Common Divisor (GCD) is 1.
- will show the following language is in P:

  RELPRIME = {<x, y> : x and y are relatively prime}
- what is the running time of the algorithm that tries all divisors up to min{x, y}?

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Euclid's Algorithm

- possibly earliest recorded algorithm

on input <x, y>:
- repeat until y = 0
  - set x = x mod y
  - swap x, y
- x is the GCD(x, y). If x = 1, accept; otherwise reject

Example run on input <10, 22>:

x, y = 10, 22

x, y = 22, 10

x, y = 10, 2

x, y = 2, 0

reject

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Euclid's Algorithm

- possibly earliest recorded algorithm

on input <x, y>:
- repeat until y = 0
  - set x = x mod y
  - swap x, y
- x is the GCD(x, y). If x = 1, accept; otherwise reject

Example run on input <24, 5>:

x, y = 24, 5

x, y = 5, 4

x, y = 4, 1

x, y = 1, 0

accept

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