CS151
Complexity Theory

Lecture 9

April 27, 2004

Outline

- The Nisan-Wigderson generator
- Error correcting codes from polynomials
- Turning worst-case hardness into average-case hardness

Hardness vs. randomness

- We have shown:

  If one-way permutations exist then

  $$\mathbf{BPP} \subset \cap_{\delta>0} \mathbf{TIME}(2^{n^{\delta}}) \subsetneq \mathbf{EXP}$$

- simulation is better than brute force, but just barely
- stronger assumptions on difficulty of inverting OWF lead to better simulations…

Hardness vs. randomness

- We will show:

  If **E** requires exponential size circuits then **BPP = P**

  by building a different generator from different assumptions.

$$\mathbf{E} = \cup_k \mathbf{DTIME}(2^{kn})$$

Hardness vs. randomness

- BMY: for every $\delta > 0$, G^{δ} is a PRG with

  seed length $\mathbf{t} = m^{\delta}$

  output length $\mathbf{m}$

  error $\boldsymbol{\varepsilon} < 1/m^d$ (all d)

  fooling size $\mathbf{s} = m^e$ (all e)

  running time m^c

- running time of simulation dominated by 2^t

Hardness vs. randomness

- To get BPP = P, would need $t = O(\log m)$
- BMY building block is one-way-permutation:

  $$f:\{0,1\}^t \rightarrow \{0,1\}^t$$

- required to fool circuits of size m^e (all e)
- with these settings a circuit has time to invert f by brute force!

  can't get BPP = P with this type of PRG

Hardness vs. randomness

- BMY pseudo-random generator:
  - one generator fooling all poly-size bounds
  - one-way-permutation is hard function
  - implies hard function in **NP** $\cap$ **coNP**
- New idea (Nisan-Wigderson):
  - for each poly-size bound, one generator
  - hard function allowed to be in

$$\mathbf{E} = \cup_k \mathbf{DTIME}(2^{kn})$$

Comparison

	BMY: $\forall$ $\delta > 0$ PRG G^δ	NW: PRG G
seed length	$t = m^\delta$	$t = O(\log m)$
running time	$t^c m$	m^c
output length	m	m
error	$\varepsilon < 1/m^d$ (all d)	$\varepsilon < 1/m$
fooling size	$s = m^e$ (all e)	$s = m$

NW PRG

- NW: for fixed constant δ, $G = \{G_n\}$ with

seed length	$t = O(\log n)$	$t = O(\log m)$
running time	n^c	m^c
output length	$m = n^\delta$	m
error	$\varepsilon < 1/m$	
fooling size	$s = m$	

- Using this PRG we obtain **BPP = P**
  - to fool size n^k use $G_{n^{k/\delta}}$
  - running time $O(n^k + n^{ck/\delta})2^t = \text{poly}(n)$

NW PRG

- First attempt: build PRG assuming **E** contains **unapproximable** functions

Definition: The function family

$$f = \{f_n\},\ f_n:\{0,1\}^n \to \{0,1\}$$

is s(n)-unapproximable if for every family of size s(n) circuits $\{C_n\}$:

$$\Pr_x[C_n(x) = f_n(x)] \le ½ + 1/s(n).$$

One bit

- Suppose $f = \{f_n\}$ is s(n)-unapproximable, for $s(n) = 2^{\Omega(n)}$, and in **E**
- a "1-bit" generator family $G = \{G_n\}$:

$$G_n(y) = y \circ f_{\log n}(y)$$

- Idea: if not a PRG then exists a predictor that computes $f_{\log n}$ with better than ½ + 1/s(log n) agreement; contradiction.

One bit

- Suppose $f = \{f_n\}$ is s(n)-unapproximable, for $s(n) = 2^{\delta n}$, and in **E**
- a "1-bit" generator family $G = \{G_n\}$:

$$G_n(y) = y \circ f_{\log n}(y)$$

  - seed length $t = \log n$
  - output length $m = \log n + 1$ (want n^δ)
  - fooling size $s \approx s(\log n) = n^\delta$
  - running time n^c
  - error $\varepsilon \approx 1/s(\log n) = 1/n^\delta < 1/m$

Many bits

- Try outputting many evaluations of f:

  $G(y) = f(b_1(y))\circ f(b_2(y))\circ\ldots\circ f(b_m(y))$

- Seems that a predictor must evaluate $f(b_i(y))$ to predict i-th bit

- Does this work?

Many bits

- Try outputting many evaluations of f:

  $G(y) = f(b_1(y))\circ f(b_2(y))\circ\ldots\circ f(b_m(y))$

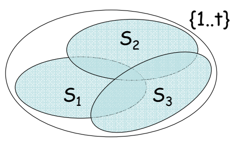
- predictor might notice correlations without having to compute f

- but, more subtle argument works for a specific choice of $b_1\ldots b_m$

Nearly-Disjoint Subsets

Definition: $S_1,S_2,\ldots,S_m \subset \{1\ldots t\}$ is an (h, a) design if

- for all i, $|S_i| = h$
- for all $i \neq j$, $|S_i \cap S_j| \leq a$



Nearly-Disjoint Subsets

Lemma: for every $\varepsilon > 0$ and $m < n$ can in poly(n) time construct an

$(h = \log n, a = \varepsilon\log n)$ design

$S_1,S_2,\ldots,S_m \subset \{1\ldots t\}$ with $t = O(\log n)$.

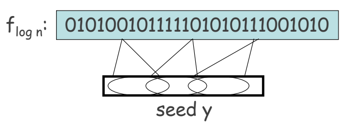
Nearly-Disjoint Subsets

- Proof sketch:
  - pick random (log n)-subset of {1…t}
  - set $t = O(\log n)$ so that expected overlap with a fixed S_i is $\varepsilon\log n/2$
  - probability overlap with S_i is $> \varepsilon\log n$ is at most 1/n
  - union bound: some subset has required small overlap with *all* S_i picked so far…
  - find it by exhaustive search; repeat n times.

The NW generator

- $f \in$ **E** s(n)-unapproximable, for $s(n) = 2^{\delta n}$
- $S_1,\ldots,S_m \subset \{1\ldots t\}$ (log n, $a = \delta\log n/3$) design with $t = O(\log n)$

  $G_n(y)=f_{\log n}(y_{|S_1})\circ f_{\log n}(y_{|S_2})\circ\ldots\circ f_{\log n}(y_{|S_m})$



The NW generator

Theorem (Nisan-Wigderson): $G=\{G_n\}$ is a pseudo-random generator with:

- seed length $\mathbf{t} = O(\log n)$
- output length $\mathbf{m} = n^{\delta/3}$
- running time n^c
- fooling size $\mathbf{s} = m$
- error $\boldsymbol{\varepsilon} = 1/m$

The NW generator

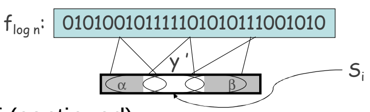
- Proof:
  - assume does not **ε**-pass statistical test $C = \{C_m\}$ of size s:

    $|Pr_x[C(x) = 1] – Pr_y[C(G_n(y)) = 1]| > \varepsilon$
  - can transform this **distinguisher** into a **predictor** P of size s' = s + O(m):

    $Pr_y[P(G_n(y)_{1\ldots i-1}) = G_n(y)_i] > ½ + \varepsilon/m$

The NW generator

$G_n(y)=f_{\log n}(y_{|S_1})\circ f_{\log n}(y_{|S_2})\circ\ldots\circ f_{\log n}(y_{|S_m})$



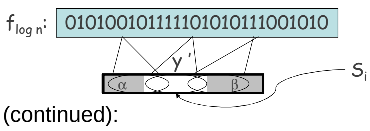
- Proof (continued):

  $Pr_y[P(G_n(y)_{1\ldots i-1}) = G_n(y)_i] > ½ + \varepsilon/m$
  - fix bits outside of S_i to preserve advantage:

  $Pr_{y'}[P(G_n(\alpha y'\beta)_{1\ldots i-1}) = G_n(\alpha y'\beta)_i] > ½ + \varepsilon/m$

The NW generator

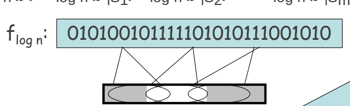
$G_n(y)=f_{\log n}(y_{|S_1})\circ f_{\log n}(y_{|S_2})\circ\ldots\circ f_{\log n}(y_{|S_m})$

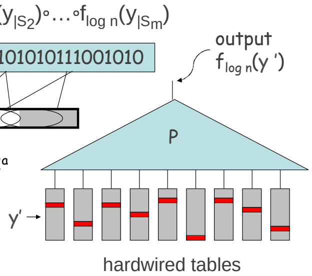


- Proof (continued):
  - $G_n(\alpha y'\beta)_i$ is exactly $f_{\log n}(y')$
  - for j ≠ i, as vary y', $G_n(\alpha y'\beta)_i$ varies over 2^a values!
  - hard-wire up to (m-1) tables of 2^a values to provide $G_n(\alpha y'\beta)_{1\ldots i-1}$

The NW generator

$G_n(y)=f_{\log n}(y_{|S_1})\circ f_{\log n}(y_{|S_2})\circ\ldots\circ f_{\log n}(y_{|S_m})$





- size $s + O(m) + (m-1)2^a$ $< s(\log n) = n^\delta$
- advantage $\varepsilon/m=1/m^2$ $> 1/s(\log n) = n^{-\delta}$
- contradiction

Worst-case vs. Average-case

Theorem (NW): if **E** contains $2^{\Omega(n)}$-unapproximable functions then **BPP** = **P**.

- How reasonable is unapproximability assumption?
- Hope: obtain **BPP** = **P** from worst-case complexity assumption
  - try to fit into existing framework without new notion of "unapproximability"

Worst-case vs. Average-case

Theorem (Impagliazzo-Wigderson, Sudan-Trevisan-Vadhan)

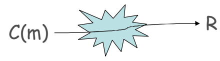
If **E** contains functions that require size $2^{\Omega(n)}$ circuits, then **E** contains $2^{\Omega(n)}$–unapp-roximable functions.

- Proof:
  - main tool: error correcting code

Error-correcting codes

- Error Correcting Code (ECC):

  $C:\Sigma^k \rightarrow \Sigma^n$
- message $m \in \Sigma^k$
- received word R
  - C(m) with some positions corrupted

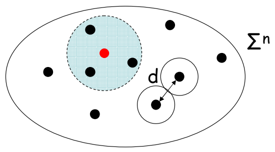


- if not too many errors, can decode: D(R) = m
- parameters of interest:
  - rate: k/n
  - distance:

  $d = \min_{m \neq m'} \Delta(C(m), C(m'))$

Distance and error correction

- C is an ECC with distance d
- can **uniquely** decode from up to $\lfloor d/2 \rfloor$ errors



Distance and error correction

- can find **short list** of messages (one correct) after closer to d errors!

Theorem (Johnson): a binary code with distance $(\frac{1}{2} - \delta^2)n$ has at most $O(1/\delta^2)$ codewords in any ball of radius $(\frac{1}{2} - \delta)n$.

Example: Reed-Solomon

- alphabet $\Sigma = \mathbf{F}_q$: field with q elements
- message $m \in \Sigma^k$
- polynomial of degree at most k-1

  $p_m(x) = \Sigma_{i=0\ldots k-1} m_i x^i$
- codeword $C(m) = (p_m(x))_{x \in \mathbf{F}_q}$
- rate = k/q

Example: Reed-Solomon

- Claim: distance d = q – k + 1
  - suppose $\Delta(C(m), C(m')) < q - k + 1$
  - then there exist polynomials $p_m(x)$ and $p_{m'}(x)$ that agree on more than k-1 points in $\mathbf{F}_q$
  - polnomial $p(x) = p_m(x) - p_{m'}(x)$ has more than k-1 zeros
  - but degree at most k-1…
  - contradiction.

Example: Reed-Muller

- Parameters: t (dimension), h (degree)
- alphabet $\Sigma = \mathbf{F}_q$: field with q elements
- message $m \in \Sigma^k$
- multivariate polynomial of total degree at most h:

$$p_m(x) = \Sigma_{i=0\ldots k-1} m_i M_i$$

$\{M_i\}$ are all monomials of degree ≤ h

Example: Reed-Muller

- M_i is monomial of total degree h
  - e.g. $x_1^2x_2x_4^3$
  - need # monomials (h+t choose t) > k
- codeword $C(m) = (p_m(x))_{x \in (\mathbf{F}_q)^t}$
- rate = k/q^t
- Claim: distance $d = (1 - h/q)q^t$
  - proof: Schwartz-Zippel: polynomial of degree h can have at most h/q fraction of zeros

Codes and hardness

- Reed-Solomon (RS) and Reed-Muller (RM) codes are efficiently encodable
- efficient unique decoding?
  - yes (classic result)
- efficient **list**-decoding?
  - yes (recent result: Sudan. On problem set.)

Codes and Hardness

- Use for worst-case to average case:

truth table of $f:\{0,1\}^{\log k} \rightarrow \{0,1\}$

(worst-case hard)

m: 0 1 1 0 0 0 1 0

truth table of $f':\{0,1\}^{\log n} \rightarrow \{0,1\}$

(average-case hard)

C(m): 0 1 1 0 0 0 1 0 0 0 0 1 0

Codes and Hardness

- if n = poly(k) then

$f \in \mathbf{E}$ implies $f' \in \mathbf{E}$

- Want to be able to prove:

if f' is s'-approximable,

then f is computable by a

size s = poly(s') circuit

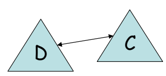
Codes and Hardness

- Key: circuit C that approximates f' implicitly gives received word R

R: 0 0 1 0 1 0 1 0 0 0 1 0 0

C(m): 0 1 1 0 0 0 1 0 0 0 0 1 0

- Decoding procedure D "computes" f exactly



• Requires special notion of efficient decoding