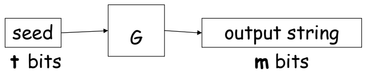
CS151
Complexity Theory

Lecture 8

April 22, 2004

Derandomization

- Goal: try to simulate BPP is subexponential time (or better)
- use Pseudo-Random Generator (PRG):



- often: PRG "good" if it passes (ad-hoc) statistical tests

Derandomization

- ad-hoc tests not good enough to prove BPP has non-trivial simulations
- Our requirements:
  - G is efficiently computable
  - "stretches" **t** bits into **m** bits
  - "fools" small circuits: for all circuits C of size at most **s**:

$$|Pr_y[C(y) = 1] – Pr_z[C(G(z)) = 1]| \leq \varepsilon$$

Simulating **BPP** using PRGs

- Recall: $L \in$ **BPP** implies exists p.p.t.TM M

  $x \in L \Rightarrow Pr_y[M(x,y) \text{ accepts}] \geq 2/3$

  $x \notin L \Rightarrow Pr_y[M(x,y) \text{ rejects}] \geq 2/3$
- given an input x:
  - convert M into circuit C(x, y)
  - simplification: pad y so that |C| = |y| = m
- hardwire input x to get circuit C_x

  $Pr_y[C_x(y) = 1] \geq 2/3$ ("yes")

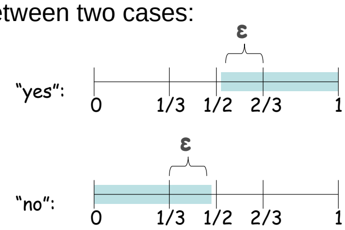
  $Pr_y[C_x(y) = 1] \leq 1/3$ ("no")

Simulating **BPP** using PRGs

- Use a PRG G with
  - output length **m**
  - seed length **t** « m
  - error ε < 1/6
  - fooling size **s** = m
- Compute $Pr_z[C_x(G(z)) = 1]$ exactly
  - evaluate $C_x(G(z))$ on every seed $z \in \{0,1\}^t$
- running time $(O(m)+(\text{time for G}))2^t$

Simulating **BPP** using PRGs

- knowing $Pr_z[C_x(G(z)) = 1]$, can distinguish between two cases:



Blum-Micali-Yao PRG

- Initial goal: for all $1 > \delta > 0$, we will build a family of PRGs $\{G_m\}$ with:

  output length $\mathbf{m}$ | fooling size $\mathbf{s} = m$
  seed length $\mathbf{t} = m^{\delta}$ | running time m^c
  error $\boldsymbol{\varepsilon} < 1/6$

- implies: $\mathbf{BPP} \subset \cap_{\delta>0} \mathbf{TIME}(2^{n^{\delta}}) \subsetneq \mathbf{EXP}$
- Why? simulation runs in time

  $O(m+m^c)(2^{m^{\delta}}) = O(2^{m^{2\delta}}) = O(2^{n^{2k\delta}})$

Blum-Micali-Yao PRG

- PRGs of this type imply existence of one-way-functions
  - we'll use widely believed cryptographic assumptions

Definition: One Way Function (OWF): function family $f = \{f_n\}$, $f_n:\{0,1\}^n \to \{0,1\}^n$

- f_n computable in poly(n) time
- for every family of poly-size circuits $\{C_n\}$

  $\Pr_x[C_n(f_n(x)) \in f_n^{-1}(f_n(x))] \leq \varepsilon(n)$
- $\varepsilon(n) = o(n^c)$ for all c

Blum-Micali-Yao PRG

- believe one-way functions exist
  - e.g. integer multiplication, discrete log, RSA (w/ minor modifications)

Definition: One Way Permutation: OWF in which f_n is 1-1

- can simplify "$\Pr_x[C_n(f_n(x)) \in f_n^{-1}(f_n(x))] \leq \varepsilon(n)$" to

  $\Pr_y[C_n(y) = f_n^{-1}(y)] \leq \varepsilon(n)$

First attempt

- attempt at PRG from OWF f:
  - $t = m^{\delta}$
  - $Y_0 \in \{0,1\}^t$
  - $y_i = f_t(y_{i-1})$
  - $G(y_0) = y_{k-1}y_{k-2}y_{k-3}\ldots y_0$
  - $k = m/t$
- computable in time at most

  $kt^c < m^c = m^c$

First attempt

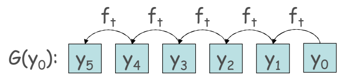
- output is "**unpredictable**":
  - no poly-size circuit C can output y_{i-1} given $y_{k-1}y_{k-2}y_{k-3}\ldots y_i$ with non-negl. success prob.
  - if C could, then given y_i can compute $y_{k-1}, y_{k-2}, \ldots, y_{i+2}, y_{i+1}$ and feed to C
  - result is poly-size circuit to compute

    $y_{i-1} = f_t^{-1}(y_i)$ from y_i
  - note: we're using that f_t is 1-1

First attempt

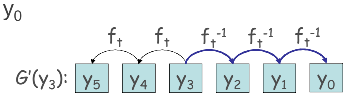
attempt:

- $Y_0 \in \{0,1\}^t$
- $y_i = f_t(y_{i-1})$



- $G(y_0) =$ $y_{k-1}y_{k-2}y_{k-3}\ldots y_0$

same distribution!



First attempt

- one problem:
  - hard to compute y_{i-1} from y_i
  - but might be easy to compute single bit (or several bits) of y_{i-1} from y_i
  - could use to build small circuit C that distinguishes G's output from uniform distribution on $\{0,1\}^m$

First attempt

- second problem
  - we don't know if "unpredictability" given a prefix is sufficient to meet fooling requirement:

$$|Pr_y[C(y) = 1] – Pr_z[C(G(z)) = 1]| \le \boldsymbol{\varepsilon}$$

Hard bits

- If $\{f_n\}$ is one-way permutation we know:
  - no poly-size circuit can compute $f_n^{-1}(y)$ from y with non-negligible success probability

$$Pr_y[C_n(y) = f_n^{-1}(y)] \le \varepsilon'(n)$$

- We want to identify a single bit position j for which:
  - no poly-size circuit can compute $(f_n^{-1}(x))_j$ from x with non-negligible advantage over a coin flip

$$Pr_y[C_n(y) = (f_n^{-1}(y))_j] \le ½ + \varepsilon(n)$$

Hard bits

- For some specific functions f we know of such a bit position j
- More general:

function $h_n:\{0,1\}^n \rightarrow \{0,1\}$

rather than just a bit position j.

Hard bits

Definition: hard bit for $g = \{g_n\}$ is family $h = \{h_n\}$, $h_n:\{0,1\}^n \rightarrow \{0,1\}$ such that if circuit family $\{C_n\}$ of size s(n) achieves:

$$Pr_x[C_n(x) = h_n(g_n(x))] \ge ½ + \varepsilon(n)$$

then there is a circuit family $\{C'_n\}$ of size s'(n) that achieves:

$$Pr_x[C'_n(x) = g_n(x)] \ge \varepsilon'(n)$$

with:
- $\varepsilon'(n) = (\varepsilon(n)/n)^{O(1)}$
- $s'(n) = (s(n)n/\varepsilon(n))^{O(1)}$

Goldreich-Levin

- To get a generic hard bit, first need to modify our one-way permutation
- Define $f'_n :\{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^{2n}$ as:

$$f'_n(x,y) = (f_n(x), y)$$

Goldreich-Levin

- Two observations: $f'_n(x,y) = (f_n(x), y)$
  - f' is a permutation if f is
  - if circuit C_n achieves
    $Pr_{x,y}[C_n(x,y) = f'^{-1}_n(x,y)] \geq \varepsilon(n)$
    then for some y^*
    $Pr_x[C_n(x,y^*)=f'^{-1}_n(x,y^*)=(f_n^{-1}(x), y^*)] \geq \varepsilon(n)$
    and so f' is a one-way permutation if f is.

Goldreich-Levin

- The Goldreich-Levin function:
  $$GL_{2n} : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$$
  is defined by:
  $$GL_{2n}(x,y) = \oplus_{i:y_i = 1} x_i$$
  - parity of subset of bits of x selected by 1's of y
  - inner-product of n-vectors x and y in GF(2)

Theorem (G-L): for every function f, GL is a hard bit for f'. (proof: problem set)

Distinguishers and predictors

- Distribution D on $\{0,1\}^n$
- D $\boldsymbol{\varepsilon}$-passes statistical tests of size s if for all circuits of size s:
  $$|Pr_{y\leftarrow U_n}[C(y) = 1] - Pr_{y \leftarrow D}[C(y) = 1]| \leq \boldsymbol{\varepsilon}$$
  - circuit violating this is sometimes called an efficient "distinguisher"

Distinguishers and predictors

- D $\boldsymbol{\varepsilon}$-passes prediction tests of size s if for all circuits of size s:
  $$Pr_{y\leftarrow D}[C(y_{1,2,\ldots,i-1}) = y_i] \leq ½ + \boldsymbol{\varepsilon}$$
  - circuit violating this is sometimes called an efficient "predictor"
- predictor seems stronger
- Yao showed essentially the same!
  - important result and proof ("hybrid argument")

Distinguishers and predictors

Theorem (Yao): if a distribution D on $\{0,1\}^n$ (ε/n)-passes all prediction tests of size s, then it ε-passes all statistical tests of size $s' = s - O(n)$.

Distinguishers and predictors

- Proof:
  - idea: proof by contradiction
  - given a size s' distinguisher C:
    $$|Pr_{y\leftarrow U_n}[C(y) = 1] - Pr_{y \leftarrow D}[C(y) = 1]| > \varepsilon$$
  - produce size s predictor P:
    $$Pr_{y\leftarrow D}[P(y_{1,2,\ldots,i-1}) = y_i] > ½ + \varepsilon/n$$
  - work with distributions that are "hybrids" of the uniform distribution U_n and D

Distinguishers and predictors

– given a size s' distinguisher C:

$$|Pr_{y\leftarrow U_n}[C(y) = 1] – Pr_{y \leftarrow D}[C(y) = 1]| > \varepsilon$$

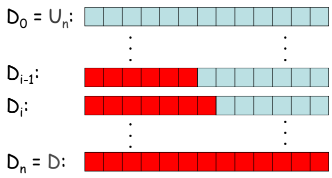
– define n+1 hybrid distributions

– hybrid distribution D_i:
  - sample $b = b_1b_2\ldots b_n$ from D
  - sample $r = r_1r_2\ldots r_n$ from U_n
  - output:

$$b_1b_2\ldots b_i\, r_{i+1}r_{i+2}\ldots r_n$$

Distinguishers and predictors

- Hybrid distributions:



Distinguishers and predictors

– Define: $p_i = Pr_{y\leftarrow D_i}[C(y) = 1]$

– Note: $p_0=Pr_{y\leftarrow U_n}[C(y)=1]$; $p_n=Pr_{y\leftarrow D}[C(y)=1]$

– by assumption: $\varepsilon < |p_n – p_0|$

– triangle inequality: $|p_n – p_0| \le \Sigma_{1 \le i \le n}|p_i – p_{i-1}|$

– there must be some i for which

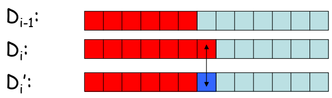
$$|p_i – p_{i-1}| > \varepsilon/n$$

– WLOG assume $p_i – p_{i-1} > \varepsilon/n$
  - can invert output of C if necessary

Distinguishers and predictors

– define distribution D_i' to be D_i with i-th bit flipped

– $p_i' = Pr_{y\leftarrow D_i'}[C(y) = 1]$



– notice:

$$D_{i-1} = (D_i + D_i')/2 \qquad p_{i-1} = (p_i + p_i')/2$$

Distinguishers and predictors

- randomized predictor P' for i^{th} bit:

– input: $u = y_1y_2\ldots y_{i-1}$

– flip a coin: $\mathbf{d} \in \{0, 1\}$

– $w = w_{i+1}w_{i+2}\ldots w_n \leftarrow U_{n-i}$

– evaluate C(u**d**w)

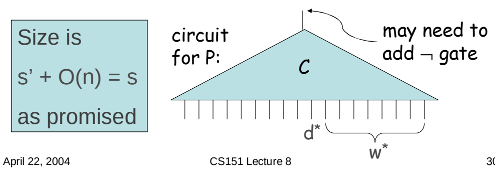
– if 1, output d; if 0, output $\neg$d

Claim:

$$Pr_{y \leftarrow D,d,w\leftarrow U_{n-i}}[P'(y_1\ldots_{i-1}) = y_i] > ½ + \varepsilon/n.$$

Distinguishers and predictors

- P' is randomized procedure
- there must be some fixing of its random bits d, w that preserves the success prob.
- final predictor P has d^* and w^* hardwired:



Distinguishers and predictors

- Proof of claim:

  $Pr_{y \leftarrow D, d, w \leftarrow U_{n-i}}[P'(y_1 \ldots_{i-1}) = y_i] =$

  $Pr[y_i = d \mid C(u,d,w) = 1]Pr[C(u,d,w) = 1]$

  $+ Pr[y_i = \neg d \mid C(u,d,w) = 0]Pr[C(u,d,w) = 0]$

  $= Pr[y_i = d \mid C(u,d,w) = 1](p_{i-1})$

  $+ Pr[y_i = \neg d \mid C(u,d,w) = 0](1 - p_{i-1})$

Distinguishers and predictors

– Observe:

$Pr[y_i = d \mid C(u,d,w) = 1]$

$= Pr[C(u,d,w) = 1 \mid y_i = d]Pr[y_i=d] / Pr[C(u,d,w) = 1]$

$= p_i/(2p_{i-1})$

$Pr[y_i = \neg d \mid C(u,d,w) = 0]$

$= Pr[C(u,d,w) = 0 \mid y_i= \neg d]Pr[y_i=\neg d] / Pr[C(u,d,w) = 0]$

$= (1 – p_i') / 2(1 - p_{i-1})$

Distinguishers and predictors

- Success probability:

  $Pr[y_i=d|C(u,d,w)=1](p_{i-1}) + Pr[y_i=\neg d|C(u,d,w)=0](1-p_{i-1})$
- We know:
  - $Pr[y_i = d \mid C(u,d,w) = 1] = p_i/(2p_{i-1})$
  - $Pr[y_i = \neg d \mid C(u,d,w) = 0] = (1 - p_i')/2(1 - p_{i-1})$
  - $p_{i-1} = (p_i + p_i')/2$
  - $p_i – p_{i-1} > \varepsilon/n$
- Conclude:

  $Pr[P'(y_1 \ldots_{i-1}) = y_i] = ½ + (p_i - p_i')/2 = ½ + p_i – p_{i-1}$

  $> ½ + \varepsilon/n.$

The BMY Generator

- Recall goal: for all $1 > \delta > 0$, family of PRGs $\{G_m\}$ with

  output length **m** — fooling size **s** = m

  seed length **t =** m^δ — running time m^c

  error **ε** < 1/6

- If one way permutations exist then WLOG there is an $f = \{f_n\}$ with a hard bit $h = \{h_n\}$

The BMY Generator

- Generator $G^\delta = \{G^\delta_m\}$:
  - $t = m^\delta$
  - $Y_0 \in \{0,1\}^t$
  - $y_i = f_t(y_{i-1})$
  - $b_i = h_t(y_i)$
  - $G^\delta(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

The BMY Generator

<u>Theorem</u> (BMY): for every $\delta > 0$, and all d, e, G^δ is a PRG with

error **ε** < $1/m^d$

fooling size **s** = m^e

running time m^c

- Note: stronger than we needed
  - sufficient to have **ε** < 1/6; **s** = m

The BMY Generator

Generator $G^\delta = \{G^\delta_m\}$:
- $t = m^\delta$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^\delta_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

- Proof:
  - computable in time at most

    $mt^c < m^{c+1}$
  - assume G^δ does not $(1/m^d)$-pass statistical test $C = \{C_m\}$ of size m^e:

    $|\Pr_{y\leftarrow U}[C(y) = 1] - \Pr_{z\leftarrow D}[C(z) = 1]| > 1/m^d$

The BMY Generator

Generator $G^\delta = \{G^\delta_m\}$:
- $t = m^\delta$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^\delta_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

- can transform this **distinguisher** into a **predictor** P of size $m^e + O(m)$:

  $\Pr_y[P(b_{m-1}\ldots b_{m-i}) = b_{m-i-1}] > ½ + 1/m^{d-1}$

The BMY Generator

Generator $G^\delta = \{G^\delta_m\}$:
- $t = m^\delta$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^\delta_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

- a procedure to compute $h_t(f_t^{-1}(y))$
  - set $y_{m-i} = y$; $b_{m-i} = h_t(y_{m-i})$
  - compute y_j, b_j for $j = m-i+1, m-i+2\ldots, m-1$ as above
  - evaluate $P(b_{m-1}b_{m-2}\ldots b_{m-i})$
  - f a permutation implies $b_{m-1}b_{m-2}\ldots b_{m-i}$ distributed as (prefix of) output of generator:

    $\Pr_y[P(b_{m-1}b_{m-2}\ldots b_{m-i}) = b_{m-i-1}] > ½ + 1/m^{d-1}$

The BMY Generator

Generator $G^\delta = \{G^\delta_m\}$:
- $t = m^\delta$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^\delta_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

$\Pr_y[P(b_{m-1}b_{m-2}\ldots b_{m-i}) = b_{m-i-1}] > ½ + 1/m^{d-1}$

- What is b_{m-i-1}?

  $b_{m-i-1} = h_t(y_{m-i-1}) = h_t(f_t^{-1}(y_{m-i})) = h_t(f_t^{-1}(y))$
- We have described a family of polynomial-size circuits that computes $h_t(f_t^{-1}(y))$ from y with success greater than ½ + 1/poly(m)
- Contradiction.

The BMY Generator