CS151
Complexity Theory

Lecture 7

April 20, 2004

Outline

- 3 examples of the power of randomness
  - communication complexity
  - polynomial identity testing
  - complexity of finding unique solutions
- randomized complexity classes
- Adelman's Theorem

Communication complexity

two parties: Alice and Bob

function $f:\{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$

Alice holds $x \in \{0,1\}^n$; Bob holds $y \in \{0,1\}^n$

- Goal: compute f(x, y) while communicating as few bits as possible between Alice and Bob
- count number of bits exchanged (computation free)
- at each step: one party sends bits that are a function of held input and received bits so far

Communication complexity

- simple function (equality):

  EQ(x, y) = 1 iff x = y

- simple protocol:
  - Alice sends x to Bob (n bits)
  - Bob sends EQ(x, y) to Alice (1 bit)
  - total: n + 1 bits
  - (works for any predicate f)

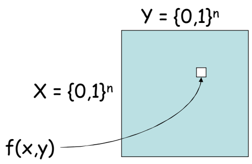
Communication complexity

- Can we do better?
  - deterministic protocol?
  - probabilistic protocol?
    - at each step: one party sends bits that are a function of held input and received bits so far and the result of some coin tosses
    - required to output f(x, y) with high probability over all coin tosses

Communication complexity

Theorem: no deterministic protocol can compute EQ(x, y) while exchanging fewer than n+1 bits.

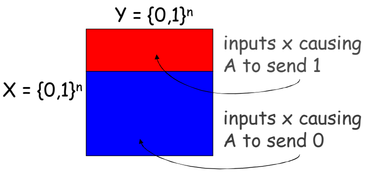
- Proof:
  - "input matrix":



Communication complexity

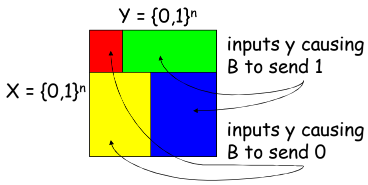
– assume without loss of generality 1 bit sent at a time

– A sends 1 bit depending only on x:



Communication complexity

– B sends 1 bit depending only on y and received bit:



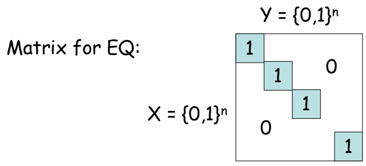
Communication complexity

– at end of protocol involving k bits of communication, matrix is partitioned into at most 2^k combinatorial rectangles

– bits sent in protocol are the same for every input (x, y) in given rectangle

– conclude: f(x,y) must be constant on each rectangle

Communication complexity



– any partition into combinatorial rectangles with constant f(x,y) must have $2^n + 1$ rectangles

– protocol that exchanges ≤ n bits can only create 2^n rectangles, so must exchange at least n+1 bits.

Communication complexity

- protocol for EQ employing randomness?
  – Alice picks random prime p in $\{1...4n^2\}$, sends:
    - p
    - (x mod p)
  – Bob sends:
    - (y mod p)
  – players output 1 if and only if:

(x mod p) = (y mod p)

Communication complexity

– O(log n) bits exchanged

– if x = y, always correct

– if x ≠ y, incorrect if and only if:

p divides |x – y|

– # primes in range is ≥ 2n

– # primes dividing |x – y| is ≤ n

– probability incorrect ≤ 1/2

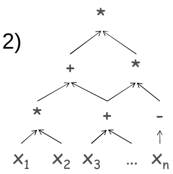
Randomness gives an exponential advantage!!

Polynomial identity testing

- Given: polynomial $p(x_1, x_2, \ldots, x_n)$ over field **F**
- Is p identically zero?
  - i.e., is $p(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{F}^n$
  - (assume |**F**| larger than degree…)
- "polynomial identity testing" because given two polynomials p, q, we can check the identity $p \equiv q$ by checking if $(p - q) \equiv 0$

Polynomial identity testing

- polynomial $p(x_1, x_2, \ldots, x_n)$ given as arithmetic circuit:
  - multiplication (fan-in 2)
  - addition (fan-in 2)
  - negation (fan-in 1)



Polynomial identity testing

- try all $|\mathbf{F}|^n$ inputs?
  - may be exponentially many
- multiply out symbolically, check that all coefficients are zero?
  - may be exponentially many coefficients
- can randomness help?
  - i.e., flip coins, allow small probability of wrong answer

Polynomial identity testing

<u>Lemma</u> (Schwartz-Zippel): Let

$$p(x_1, x_2, \ldots, x_n)$$

be a total degree d polynomial over a field **F** and let S be any subset of **F**. Then if p is not identically 0,

$$\Pr_{r_1,r_2,\ldots,r_n \in S}[\, p(r_1, r_2, \ldots, r_n) = 0] \leq d/|S|.$$

Polynomial identity testing

- Proof:
  - induction on number of variables n
  - base case: n = 1, p is univariate polynomial of degree at most d
  - at most d roots, so

$$\Pr[\, p(r_1) = 0] \leq d/|S|$$

Polynomial identity testing

- write $p(x_1, x_2, \ldots, x_n)$ as

$$p(x_1, x_2, \ldots, x_n) = \Sigma_i (x_1)^i \, p_i(x_2, \ldots, x_n)$$

- k = max. i for which $p_i(x_2, \ldots, x_n)$ not id. zero
- by induction hypothesis:

$$\Pr[\, p_k(r_2, \ldots, r_n) = 0] \leq (d-k)/|S|$$

- whenever $p_k(r_2, \ldots, r_n) \neq 0$, $p(x_1, r_2, \ldots, r_n)$ is a univariate polynomial of degree k

$$\Pr[p(r_1,r_2,\ldots,r_n)=0 \mid p_k(r_2,\ldots,r_n) \neq 0] \leq k/|S|$$

Polynomial identity testing

$\Pr[\, p_k(r_2, \ldots, r_n) = 0] \leq (d-k)/|S|$

$\Pr[p(r_1, r_2, \ldots, r_n)=0 \mid p_k(r_2, \ldots, r_n) \neq 0] \leq k/|S|$

– conclude:

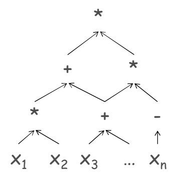
$\Pr[\, p(r_1, \ldots, r_n) = 0] \leq (d-k)/|S| + k/|S| = d/|S|$

– Note: can add these probabilities because

$\Pr[E_1] = \Pr[E_1|E_2]\Pr[E_2] + \Pr[E_1|\neg E_2]\Pr[\neg E_2]$
$\leq \Pr[E_2] + \Pr[E_1|\neg E_2]$

Polynomial identity testing

- Given: polynomial $p(x_1, x_2, \ldots, x_n)$ over field **F**
- Is p identically zero?



- Note: degree d is at most the size of input

Polynomial identity testing

- randomized algorithm: pick a subset $S \subset$ **F** of size 2d
  - pick $r_1, r_2, \ldots, r_n$ from S uniformly at random
  - if $p(r_1, r_2, \ldots, r_n) = 0$, answer "yes"
  - if $p(r_1, r_2, \ldots, r_n) \neq 0$, answer "no"
- if p identically zero, never wrong
- if not, Schwartz-Zippel ensures probability of error at most ½

Unique solutions

- a positive instance of SAT may have many satisfying assignments
- maybe the difficulty comes from not knowing which to "work on"
- if we knew # satisfying assignments was 1 or 0, could we zero in on the 1 efficiently?

Unique solutions

<u>**Question**</u>: given polynomial-time algorithm that works on SAT instances with at most 1 satisfying assignment, can we solve general SAT instances efficiently?

- Answer: yes
  - but only if "efficiently" allows randomness

Unique solutions

<u>**Theorem**</u> (Valiant-Vazirani): there is a randomized poly-time procedure that given a 3-CNF formula

$\varphi(x_1, x_2, \ldots, x_n)$

outputs a 3-CNF formula φ' such that
  - if φ is not satisfiable then φ' is not satisfiable
  - if φ is satisfiable then with probability at least 1/(8n) φ' has exactly one satisfying assignment

Unique solutions

- Proof:
  - given subset $S \subset \{1, 2, \ldots, n\}$, there exists a 3-CNF formula θ_S on $x_1, x_2, \ldots, x_n$ and additional variables such that:
    - $|\theta_S| = O(n)$
    - θ_S is satisfiable iff an even number of variables in $\{x_i\}_{i \in S}$ are true
    - for each such setting of the x_i variables, this satisfying assignment is unique
    - not difficult; details omitted

Unique solutions

- set $\varphi_0 = \varphi$
- for i = 1, 2, …, n
  - pick random subset S_i
  - set $\varphi_i = \varphi_{i-1} \wedge \theta_{S_i}$
- output random φ_i

- T = set of satisfying assignments for φ
- Claim: if |T| > 0, then

$$\Pr_{k \in \{0,1,2,\ldots,n-1\}}[2^k \leq |T| \leq 2^{k+1}] \geq 1/n$$

Unique solutions

Claim: if $2^k \leq |T| \leq 2^{k+1}$, then the probability φ_{k+2} has exactly one satisfying assignment is $\geq 1/8$

- fix $t \in T$

t = 0101 $\boxed{00101}$ 0111

t' = 1010 $\boxed{11100}$ 0101

S_i

- Pr[t "agrees with" t' on S_i] = ½
- Pr[t agrees with t' on $S_1, S_2, \ldots, S_{k+2}$] = $(½)^{k+2}$

Unique solutions

- Pr[t agrees with some t' on $S_1, \ldots, S_{k+2}$]

  $\leq (|T|-1)(½)^{k+2} < ½$
- Pr[t satisfies $S_1, S_2, \ldots, S_{k+2}$] = $(½)^{k+2}$
- Pr[t *unique* satisfying assignment of φ_{k+2}]

  $> (½)^{k+3}$

- sum over at least 2^k different $t \in T$ (disjoint events); claim follows.

Randomized complexity classes

- model: probabilistic Turing Machine
  - deterministic TM with additional read-only tape containing "coin flips"
- **BPP** (Bounded-error Probabilistic Poly-time)
  - $L \in$ **BPP** if there is a p.p.t. TM M:

    $x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq 2/3$

    $x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] \geq 2/3$
  - "p.p.t" = probabilistic polynomial time

Randomized complexity classes

- **RP** (Random Polynomial-time)
  - $L \in$ **RP** if there is a p.p.t. TM M:

    $x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq ½$

    $x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] = 1$
- **coRP** (complement of Random Polynomial-time)
  - $L \in$ **coRP** if there is a p.p.t. TM M:

    $x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] = 1$

    $x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] \geq ½$

Randomized complexity classes

These classes may capture "efficiently computable" better than **P**.

- "1/2" in **RP**, **coRP** definition unimportant
  - can replace by 1/poly(n)
- "2/3" in **BPP** definition unimportant
  - can replace by ½ + 1/poly(n)
- Why? error reduction
  - we will see simple error reduction by repetition
  - more sophisticated error reduction later

Error reduction for RP

- given L and p.p.t TM M:

$x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq \varepsilon$

$x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] = 1$

- new p.p.t TM M':
  - simulate M k/ε times, each time with independent coin flips
  - accept if **any** simulation accepts
  - otherwise reject

Error reduction

$x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq \varepsilon$

$x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] = 1$

- if $x \in L$:
  - probability a given simulation "bad" ≤ (1 – ε)
  - probability all simulations "bad" $\leq (1-\varepsilon)^{(k/\varepsilon)} \leq e^{-k}$

$\Pr_{y'}[M'(x, y') \text{ accepts}] \geq 1 - e^{-k}$

- if $x \notin L$:

$\Pr_{y'}[M'(x,y') \text{ rejects}] = 1$

Error reduction for BPP

- given L, and p.p.t. TM M:

$x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq ½ + \varepsilon$

$x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] \geq ½ + \varepsilon$

- new p.p.t. TM M':
  - simulate M k/ε^2 times, each time with independent coin flips
  - accept if **majority** of simulations accept
  - otherwise reject

Error reduction for BPP

- X_i random variable indicating "correct" outcome in i-th simulation (out of $m = k/\varepsilon^2$)
  - $\Pr[X_i = 1] \geq ½ + \varepsilon$
  - $\Pr[X_i = 0] \leq ½ - \varepsilon$
- $E[X_i] \geq ½+\varepsilon$
- $X = \Sigma_i X_i$
- $\mu = E[X] \geq (½ + \varepsilon)m$
- Chernoff: $\Pr[X \leq m/2] \leq 2^{-\Omega(\varepsilon^2 \mu)}$

Error reduction for BPP

$x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq ½ + \varepsilon$

$x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] \geq ½ + \varepsilon$

- if $x \in L$

$\Pr_{y'}[M'(x, y') \text{ accepts}] \geq 1 - (½)^{\Omega(k)}$

- if $x \notin L$

$\Pr_{y'}[M'(x,y') \text{ rejects}] \geq 1 - (½)^{\Omega(k)}$

Randomized complexity classes

One more important class:

- **ZPP** (Zero-error Probabilistic Poly-time)
  - **ZPP = RP ∩ coRP**
  - $\Pr_y$[M(x,y) outputs "fail"] ≤ ½
  - otherwise outputs correct answer

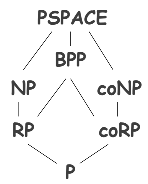
Randomized complexity classes

- We have shown:
  - polynomial identity testing is in **coRP**
  - a poly-time algorithm for detecting unique solutions to SAT implies **NP = RP**

Relationship to other classes

- all these classes contain **P**
  - they can simply ignore the tape with coin flips
- all are in **PSPACE**
  - can exhaustively try all strings y
  - count accepts/rejects; compute probability
- **RP ⊂ NP** (and **coRP ⊂ coNP**)
  - multitude of accepting computations
  - **NP** requires only one

Relationship to other classes



BPP

- How powerful is **BPP**?
- We have seen an example of a problem in **BPP** that we only know how to solve in **EXP**.

Is randomness a panacea for intractability?

BPP

- It is not known if **BPP = EXP** (or even **NEXP!**)
  - but there are strong hints that it does not
- Is there a deterministic simulation of **BPP** that does better than brute-force search?
  - yes, if allow non-uniformity

Theorem (Adelman): **BPP ⊂ P/poly**

BPP and Boolean circuits

- Proof:
  - language L $\in$ **BPP**
  - error reduction gives TM M such that
    - if $x \in L$

      $\Pr_y[M(x, y) \text{ accepts}] \geq 1 - (½)^{|x|^2}$
    - if $x \notin L$

      $\Pr_y[M(x, y) \text{ rejects}] \geq 1 - (½)^{|x|^2}$

BPP and Boolean circuits

- say "y is bad for x" if M(x,y) gives incorrect answer
- for fixed x: $\Pr_y[y \text{ is bad for } x] \leq (½)^{|x|^2}$
- $\Pr_y[y \text{ is bad for some } x] \leq 2^{|x|}(½)^{|x|^2} < 1$
- Conclude: there exists some y on which M(x, y) is always correct
- build circuit for M, hardwire this y

BPP and Boolean circuits

- Does **BPP** = **EXP** ?
- Adelman's Theorem shows:

  BPP = **EXP** implies **EXP** $\subset$ **P/poly**

If you believe that randomness is all-powerful, you must also believe that non-uniformity gives an exponential advantage.

BPP

- Next:

  further explore the relationship between

  randomness

  and

  nonuniformity
- Main tool: pseudo-random generators