CS151
Complexity Theory

Lecture 15

May 18, 2004

Outline

- **IP** = **PSPACE**
- **Arthur-Merlin** games
  - classes MA, AM
- Optimization, Approximation, and Probabilistically Checkable Proofs

Shamir's Theorem

Theorem: **IP** = **PSPACE**
- Note: **IP** $\subset$ **PSPACE**
  - enumerate all possible interactions, explicitly calculate acceptance probability
- interaction extremely powerful !
- An implication: you can interact with master player of Generalized Geography and determine if she can win from the current configuration even if you do not have the power to compute optimal moves!

Shamir's Theorem

- need to prove **PSPACE** $\subset$ **IP**
  - use same protocol as for **coNP**
  - some modifications needed

Shamir's Theorem

- protocol for QSAT
  - arithmetization step produces **arithmetic expression** p_φ:
    - $(\exists x_i)\ \varphi \rightarrow \Sigma_{x_i = 0, 1}\ p_\varphi$
    - $(\forall x_i)\ \varphi \rightarrow \prod_{x_i = 0, 1}\ p_\varphi$
  - start with QSAT formula in special form ("simple")
    - no occurrence of x_i separated by more than one "$\forall$" from point of quantification

Shamir's Theorem

- quantified Boolean expression φ is true if and only if $p_\varphi > 0$
- Problem: $\prod$'s may cause $p_\varphi > 2^{2^{|\varphi|}}$
- Solution: evaluate mod $2^n \leq q \leq 2^{3n}$
- prover sends "good" q in first round
  - "good" q is one for which p_φ mod $q > 0$
- Claim: good q exists
  - # primes in range is at least 2^n

The QSAT protocol

input: φ

Prover

$k, q, p_1(x)$ →

$p_1(x)$: remove outer Σ or $\prod$ from p_φ

← z_1

$p_2(x)$: remove outer Σ or $\prod$ from $p_\varphi[x_1 \leftarrow z_1]$

$p_2(x)$ →

← z_2

$p_3(x)$: remove outer Σ or $\prod$ from $p_\varphi[x_1 \leftarrow z_1, x_2 \leftarrow z_2]$

$p_3(x)$ →

.

.

$p_n(x)$ →

Verifier

$p_1(0)+p_1(1) = k$? or $p_1(0)p_1(1) = k$?

pick random z_1 in F_q

$p_2(0)+p_2(1)=p_1(z_1)$? or $p_2(0)p_2(1) = p_1(z_1)$?

pick random z_2 in F_q

$p_n(0)+p_n(1)=p_{n-1}(z_{n-1})$? or $p_n(0)p_n(1) = p_{n-1}(z_{n-1})$?

pick random z_n in F_q

$p_n(z_n) = p_\varphi[x_1 \leftarrow z_1,\ldots, x_n \leftarrow z_n]$

Analysis of the QSAT protocol

- Completeness:
  - if $\varphi \in$ QSAT then honest prover on previous slide will always cause verifier to accept

Analysis of the QSAT protocol

- Soundness:
  - let $p_i(x)$ be the correct polynomials
  - let $p_i^*(x)$ be the polynomials sent by (cheating) prover
  - $\varphi \notin$ QSAT $\Rightarrow 0 = p_1(0)$ +/x $p_1(1) \neq k$
  - either $p_1^*(0)$ +/x $p_1^*(1) \neq k$ (and V rejects) — φ is "simple"
  - or $p_1^* \neq p_1 \Rightarrow \Pr_{z_1}[p_1^*(z_1) = p_1(z_1)] \leq 2|\varphi|/2^n$
  - assume ($p_{i+1}(0)$ +/x $p_{i+1}(1)$=) $p_i(z_i) \neq p_i^*(z_i)$
  - either $p_{i+1}^*(0)$ +/x $p_{i+1}^*(1) \neq p_i^*(z_i)$ (and V rejects)
  - or $p_{i+1}^* \neq p_{i+1} \Rightarrow \Pr_{z_{i+1}}[p_{i+1}^*(z_{i+1}) = p_{i+1}(z_{i+1})] \leq 2|\varphi|/2^n$

Analysis of protocol

- Soundness (continued):
  - if verifier does not reject, there must be some i for which:

    $p_i^* \neq p_i$ and yet $p_i^*(z_i) = p_i(z_i)$
  - for each i, probability is $\leq 2|\varphi|/2^n$
  - union bound: probability that there exists an i for which the bad event occurs is

    $\leq 2n|\varphi|/2^n \leq \mathrm{poly}(n)/2^n << 1/3$
- Conclude: QSAT is in **IP**

Example

- Papadimitriou – pp. 475-480

$$\varphi = \forall x \exists y (x \vee y) \wedge \forall z ((x \wedge z) \vee (y \wedge \neg z)) \vee \exists w (z \vee (y \wedge \neg w))$$

$$p_\varphi = \prod_{x=0,1} \Sigma_{y=0,1}[(x + y) * \prod_{z=0,1}[(xz + y(1-z)) + \Sigma_{w=0,1}(z + y(1-w))]]$$

(p_φ = 96 but V doesn't know that yet !)

Example

$$p_\varphi = \prod_{x=0,1} \Sigma_{y=0,1}[(x + y) * \prod_{z=0,1}[(xz + y(1-z)) + \Sigma_{w=0,1}(z + y(1-w))]]$$

Round 1: (prover claims $p_\varphi > 0$)

- prover sends q = 13; claims p_φ = 96 mod 13 = 5; sends k = 5
- prover removes outermost "$\prod$"; sends

  $p_1(x) = 2x^2 + 8x + 6$
- verifier checks:

  $p_1(0)p_1(1) = (6)(16) = 96 \equiv 5 \pmod{13}$
- verifier picks randomly: $z_1 = 9$

Example

$\varphi = \forall x\exists y(x\vee y)\wedge\forall z((x\wedge z)\vee(y\wedge\neg z))\vee\exists w(z\vee(y\wedge\neg w))$

$p_\varphi = \prod_{x=0,1}\Sigma_{y=0,1}[(x + y) * \prod_{z=0,1}[(xz + y(1\text{-}z)) + \Sigma_{w=0,1}(z + y(1\text{-}w))]]$

$p_\varphi[x\leftarrow 9] = \Sigma_{y=0,1}[(9 + y) * \prod_{z=0,1}[(9z + y(1\text{-}z)) + \Sigma_{w=0,1}(z + y(1\text{-}w))]]$

May 18, 2004 CS151 Lecture 15 13

Example

$p_1(9) = \Sigma_{y=0,1}[(9 + y) * \prod_{z=0,1}[(9z + y(1\text{-}z)) + \Sigma_{w=0,1}(z + y(1\text{-}w))]]$

Round 2: (prover claims this = 6)

- prover removes outermost "Σ"; sends
  $p_2(y) = 2y^3 + y^2 + 3y$
- verifier checks:
  $p_2(0) + p_2(1) = 0 + 6 = 6 \equiv 6 \pmod{13}$
- verifier picks randomly: $z_2 = 3$

May 18, 2004 CS151 Lecture 15 14

Example

$\varphi = \forall x\exists y(x\vee y)\wedge\forall z((x\wedge z)\vee(y\wedge\neg z))\vee\exists w(z\vee(y\wedge\neg w))$

$p_\varphi = \prod_{x=0,1}\Sigma_{y=0,1}[(x + y) * \prod_{z=0,1}[(xz + y(1\text{-}z)) + \Sigma_{w=0,1}(z + y(1\text{-}w))]]$

$p_\varphi[x\leftarrow 9, y\leftarrow 3] = [(9 + 3) * \prod_{z=0,1}[(9z + 3(1\text{-}z)) + \Sigma_{w=0,1}(z + 3(1\text{-}w))]]$

May 18, 2004 CS151 Lecture 15 15

Example

$p_2(3) = [(9 + 3) * \prod_{z=0,1}[(9z + 3(1\text{-}z)) + \Sigma_{w=0,1}(z + 3(1\text{-}w))]]$

Round 3: (prover claims this = 7)

- everyone agrees expression = 12*(…)
- prover removes outermost "∏"; sends
  $p_3(z) = 8z + 6$
- verifier checks:
  $p_3(0) * p_3(1) = (6)(14) = 84;\ 12*84 \equiv 7 \pmod{13}$
- verifier picks randomly: $z_3 = 7$

May 18, 2004 CS151 Lecture 15 16

Example

$\varphi = \forall x\exists y(x\vee y)\wedge\forall z((x\wedge z)\vee(y\wedge\neg z))\vee\exists w(z\vee(y\wedge\neg w))$

$p_\varphi = \prod_{x=0,1}\Sigma_{y=0,1}[(x + y) * \prod_{z=0,1}[(xz + y(1\text{-}z)) + \Sigma_{w=0,1}(z + y(1\text{-}w))]]$

$p_\varphi[x\leftarrow 9, y\leftarrow 3, z\leftarrow 7] = 12 * [(9*7 + 3(1\text{-}7)) + \Sigma_{w=0,1}(7 + 3(1\text{-}w))]]$

May 18, 2004 CS151 Lecture 15 17

Example

$12*p_3(7) = 12 * [(9*7 + 3(1\text{-}7)) + \Sigma_{w=0,1}(7 + 3(1\text{-}w))]$

Round 4: (prover claims = 12*10)

- everyone agrees expression = 12*[6+(…)]
- prover removes outermost "Σ"; sends
  $p_4(w) = 10w + 10$
- verifier checks:
  $p_4(0)+p_4(1) = 10 + 20 = 30;\ 12*[6+30] \equiv 12*10 \pmod{13}$
- verifier picks randomly: $z_4 = 2$
- Final check:
  $12*[(9*7+3(1\text{-}7))+(7+3(1\text{-}2))] = 12*[6+p_4(2)] = 12*[6+30]$

May 18, 2004 CS151 Lecture 15 18

Arthur-Merlin Games

- **IP** permits verifier to keep coin-flips private
  - necessary feature?
  - GNI protocol breaks without it
- Arthur-Merlin game: interactive protocol in which coin-flips are public
  - Arthur (verifier) may as well just send results of coin-flips and ask Merlin (prover) to perform any computation he would have done

Arthur-Merlin Games

- Clearly **Arthur-Merlin** $\subset$ **IP**
  - "private coins are at least as powerful as public coins"
- Proof that **IP** = **PSPACE** actually shows

  PSPACE $\subset$ **Arthur-Merlin** $\subset$ **IP** $\subset$ **PSPACE**
  - "public coins are at least as powerful as private coins" !

Arthur-Merlin Games

- Delimiting # of rounds:
  - **AM[k]** = Arthur-Merlin game with k rounds, Arthur (verifier) goes first
  - **MA[k]** = Arthur-Merlin game with k rounds, Merlin (prover) goes first

Theorem: **AM[k]** (**MA[k]**) equals **AM[k]** (**MA[k]**) with perfect completeness.
- i.e., $x \in L$ implies accept with probability 1
- we will not prove

Arthur-Merlin Games

Theorem: for all constant $k \geq 2$

AM[k] = **AM[2]**.

- Proof:
  - we show **MA[2]** $\subset$ **AM[2]**
  - implies can move all of Arthur's messages to beginning of interaction:

    AMAMAM…AM = AAMMAM…AM

    … = AAA…AMMM…M

Arthur-Merlin Games

- Proof (continued):
  - given $L \in$ **MA[2]**

$x \in L \Rightarrow \exists m \Pr_r[(x, m, r) \in R] = 1$

$\Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] = 1$

order reversed

$x \notin L \Rightarrow \forall m \Pr_r[(x, m, r) \in R] \leq \varepsilon$

$\Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] \leq 2^{|m|}\varepsilon$

  - by repeating t times with independent random strings r, can make error $\varepsilon < 2^{-t}$
  - set $t = m+1$ to get $2^{|m|}\varepsilon < ½$.

MA and **AM**

- Two important classes:
  - **MA** = **MA[2]**
  - **AM** = **AM[2]**
- definitions without reference to interaction:
  - $L \in$ **MA** iff $\exists$ poly-time language R

    $x \in L \Rightarrow \exists m \Pr_r[(x, m, r) \in R] = 1$

    $x \notin L \Rightarrow \forall m \Pr_r[(x, m, r) \in R] \leq ½$
  - $L \in$ **AM** iff $\exists$ poly-time language R

    $x \in L \Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] = 1$

    $x \notin L \Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] \leq ½$

MA and **AM**

L $\in$ **AM** iff $\exists$ poly-time language R

$x \in L \Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] = 1$

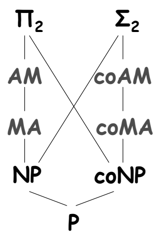
$x \notin L \Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] \leq ½$

- Relation to other complexity classes:
  - both contain **NP** (can elect to not use randomness)
  - both contained in $\prod_2$. L $\in \prod_2$ iff $\exists$ R $\in$ P for which:

    $x \in L \Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] = 1$

    $x \notin L \Rightarrow \Pr_r[\exists m\ (x, m, r) \in R] < 1$
  - so clear that **AM** $\subset \prod_2$
  - know that **MA** $\subset$ **AM**

MA and **AM**



MA and **AM**

- We know **Arthur-Merlin** = **IP**.
  - "public coins = private coins"

<u>Theorem</u> (GS): **IP[k]** $\subset$ **AM[O(k)]**
  - stronger result
  - implies for all constant k ≥ 2,

    IP[k] = **AM[O(k)]** = **AM[2]**

- So, GNI $\in$ **IP[2]** = **AM**

MA and **AM**

<u>Theorem</u>: **coNP** $\subset$ **AM** $\Rightarrow$ **PH** = **AM.**

- Proof:
  - suffices to show $\mathbf{\Sigma_2} \subset$ **AM** (and use **AM** $\subset \mathbf{\Pi_2}$)
  - L $\in \mathbf{\Sigma_2}$ iff $\exists$ poly-time language R

    $x \in L \Rightarrow \exists y\ \forall z\ (x, y, z) \in R$

    $x \notin L \Rightarrow \forall y\ \exists z\ (x, y, z) \notin R$
  - Merlin sends y
  - 1 AM exchange decides **coNP** query: $\forall z\ (x, y, z) \in R$?
  - 3 rounds; in **AM**

Back to Graph Isomorphism

- The payoff:
  - not known if GI is **NP**-complete.
  - previous Theorems:

    if GI is **NP**-complete then **PH** = **AM**
  - unlikely!
  - Proof: GI **NP**-complete $\Rightarrow$ GNI **coNP**-complete $\Rightarrow$ **coNP** $\subset$ **AM** $\Rightarrow$ **PH** = **AM**

New topic(s)

Optimization problems,

Approximation Algorithms,

and

Probabilistically Checkable Proofs

Optimization Problems

- many hard problems (especially **NP**-hard) are optimization problems
  - e.g. find *shortest* TSP tour
  - e.g. find *smallest* vertex cover
  - e.g. find *largest* clique

  - may be minimization or maximization problem
  - "opt" = value of optimal solution

Approximation Algorithms

- often happy with approximately optimal solution
  - warning: lots of heuristics
  - we want approximation algorithm with guaranteed approximation ratio of r
  - meaning: on every input x, output is guaranteed to have value

    at most r*opt for minimization

    at least opt/r for maximization

Approximation Algorithms

- Example approximation algorithm:
  - Recall:

  Vertex Cover (VC): given a graph G, what is the *smallest* subset of vertices that touch every edge?

  - **NP**-complete

Approximation Algorithms

- Approximation algorithm for VC:
  - pick an edge (x, y), add vertices x and y to VC
  - discard edges incident to x or y; repeat.
- Claim: approximation ratio is 2.
- Proof:
  - an optimal VC must include at least one endpoint of each edge considered
  - therefore 2*opt ≥ actual

Approximation Algorithms

- diverse array of ratios achievable
- some examples:
  - (min) Vertex Cover: 2
  - MAX-3-SAT (find assignment satisfying largest # clauses): 8/7
  - (min) Set Cover: ln n
  - (max) Clique: $n/\log^2 n$
  - (max) Knapsack: $(1 + \varepsilon)$ for any $\varepsilon > 0$

Approximation Algorithms

(max) Knapsack: $(1 + \varepsilon)$ for any $\varepsilon > 0$

- called Polynomial Time Approximation Scheme (PTAS)
  - algorithm runs in poly time for every fixed $\varepsilon>0$
  - poor dependence on ε allowed
- If all **NP** optimization problems had a PTAS, almost like **P** = **NP** (!)

Approximation Algorithms

- A job for complexity: How to explain failure to do better than ratios on previous slide?
  - – just like: how to explain failure to find poly-time algorithm for SAT...
  - – first guess: probably NP-hard
  - – what is needed to show this?
- "gap-producing" reduction from **NP**-complete problem L_1 to L_2

Approximation Algorithms

- "gap-producing" reduction from **NP**-complete problem L_1 to L_2