CS151
Complexity Theory

Lecture 11

May 4, 2004

Outline

- Extractors
- Trevisan's extractor
- **RL** and undirected STCONN

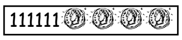
Extractors

- PRGs: can remove randomness from algorithms
  - based on unproven assumption
  - polynomial slow-down
  - not applicable in other settings
- Question: can we use "real" randomness?
  - physical source
  - imperfect – biased, correlated

Extractors

- "Hardware" side
  - what physical source?
  - ask the physicists…
- "Software" side
  - what is the minimum we need from the physical source?

Extractors

- imperfect sources:
  - "stuck bits": 

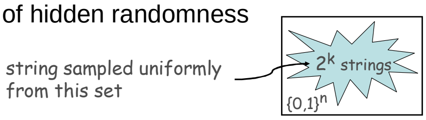
  - "correlation": 
  - "more insidious correlation": perfect squares
- there are specific ways to get independent unbiased random bits from specific imperfect physical sources

Extractors

- want to assume we don't know details of physical source
- **general model** capturing all of these?
  - yes: "min-entropy"
- **universal procedure** for all imperfect sources?
  - yes: "extractors"

Min-entropy

- General model of physical source w/ $k < n$ bits of hidden randomness

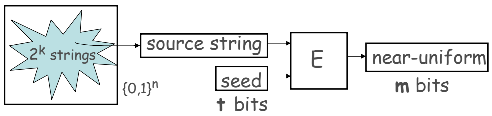


Definition: random variable X on $\{0,1\}^n$ has **min-entropy** $\min_x -\log(\Pr[X = x])$

– min-entropy k implies no string has weight more than 2^{-k}

Extractor

- Extractor: universal procedure for "purifying" imperfect source:



– E is efficiently computable
– truly random seed as "catalyst"

Extractor

"(k, ε)-extractor" ⇒ for all X with min-entropy k:

– output fools **all** circuits C:

$$|\Pr_z[C(z) = 1] - \Pr_{y,\, x \leftarrow X}[C(E(x, y)) = 1]| \leq \boldsymbol{\varepsilon}$$

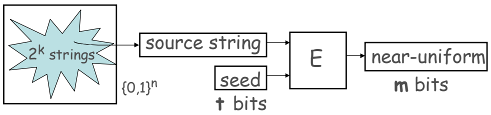
– distributions $E(X, U_t)$, U_m "**ε**-close" (L_1 dist $\leq 2\boldsymbol{\varepsilon}$)

- Notice similarity to PRGs
  – output of PRG fools all **efficient** tests
  – output of extractor fools **all** tests

Extractors

- Using extractors
  – use output in place of randomness in any application
  – alters probability of **any** outcome by at most **ε**
- Main motivation:
  – use output in place of randomness in algorithm
  – how to get truly random seed?
  – enumerate all seeds, take majority

Extractors



• Goals:	good:	best:
short seed	O(log n)	log n+O(1)
long output	$m = k^{\Omega(1)}$	$m = k+t–O(1)$
many k's	$k = n^{\Omega(1)}$	any $k = k(n)$

Extractors

- random function for E achieves best !
  – but we need **explicit** constructions
  – usually complex + technical
  – optimal extractors still open
- Trevisan Extractor:
  – insight: use NW generator with source string in place of hard function
  – this works (!!)
  – proof slightly different than NW, easier

Trevisan Extractor

- Ingredients:
  - error-correcting code

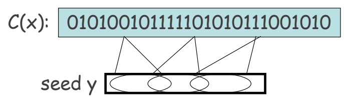
    $C:\{0,1\}^n \rightarrow \{0,1\}^{n'}$

    distance $(\frac{1}{2} - \frac{1}{4}m^{-4})n'$ blocklength $n' = \mathrm{poly}(n)$
  - $(\log n', a = \delta\log n/3)$ design: $S_1, S_2, \ldots, S_m \subset \{1\ldots t = O(\log n')\}$

$$E(x, y)=C(x)[y_{|S_1}]\circ C(x)[y_{|S_2}]\circ\ldots\circ C(x)[y_{|S_m}]$$

Trevisan Extractor

$$E(x, y)=C(x)[y_{|S_1}]\circ C(x)[y_{|S_2}]\circ\ldots\circ C(x)[y_{|S_m}]$$



Theorem (T): E is an extractor for min-entropy $k = n^\delta$, with
- output length $m = k^{1/3}$
- seed length $t = O(\log n)$
- error $\varepsilon \leq 1/m$

Trevisan Extractor

- Proof:
  - assume $X \subseteq \{0,1\}^n$
  - assume fails to ε-pass statistical test C

    $|\Pr_z[C(z) = 1] - \Pr_{x\in X, y}[C(E(x, y)) = 1]| > \varepsilon$
  - **distinguisher** C $\Rightarrow$ **predictor** P:

    $\Pr_{x\in X, y}[P(E(x, y)_{1\ldots i-1})=E(x, y)_i] > \frac{1}{2} + \varepsilon/m$

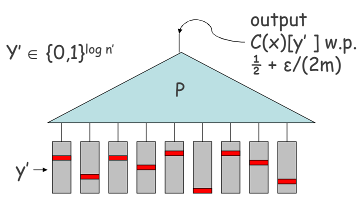
Trevisan Extractor

- Proof (continued):
  - for at least $\varepsilon/2$ of $x \in X$ we have:

    $\Pr_y[P(E(x, y)_{1\ldots i-1})=E(x, y)_i] > \frac{1}{2} + \varepsilon/(2m)$
  - fix bits w outside of S_i to preserve advantage

    $\Pr_{y'}[P(E(x; wy')_{1\ldots i-1})=C(x)[y']\,] > \frac{1}{2} + \varepsilon/(2m)$
  - as vary y', for $j \neq i$, j-th bit of $E(x; wy')$ varies over only 2^a values
  - build up to (m-1) tables of 2^a values to supply $E(x; wy')_{1\ldots i-1}$

Trevisan Extractor



Trevisan Extractor

- Proof (continued):
  - (m-1) tables of size 2^a constitute a **description** of a string that has $\frac{1}{2} + \varepsilon/(2m)$ agreement with C(x)
  - # of $x \in X$ with such a description?
    - $\exp((m-1)2^a) = \exp(n^{\delta 2/3}) = \exp(k^{2/3})$ strings
    - Johnson Bound: each string accounts for at most $O(m^4)$ x's
    - total #: $O(m^4)\exp(k^{2/3}) << 2^k(\varepsilon/2)$
    - contradiction

Strong error reduction

- $L \in$ **BPP** if there is a p.p.t. TM M:
  - $x \in L \Rightarrow Pr_y[M(x,y) \text{ accepts}] \geq 2/3$
  - $x \notin L \Rightarrow Pr_y[M(x,y) \text{ rejects}] \geq 2/3$
- Want:
  - $x \in L \Rightarrow Pr_y[M(x,y) \text{ accepts}] \geq 1 - 2^{-k}$
  - $x \notin L \Rightarrow Pr_y[M(x,y) \text{ rejects}] \geq 1 - 2^{-k}$
- We saw: repeat O(k) times
  - n = O(k)·|y| random bits; **2^{n-k} bad strings**

Strong error reduction

- Better:
  - E extractor for $k = |y|^3 = n^\delta$, $\varepsilon < 1/6$
  - pick random $w \in \{0,1\}^n$, run M(x, E(w, z)) for all $z \in \{0,1\}^t$, take majority
  - call w "bad" if $maj_z M(x, E(w, z))$ incorrect
    - $|Pr_z[M(x,E(w,z))=b] - Pr_y[M(x,y)=b]| \geq 1/6$
  - extractor property: at most 2^k bad w
  - n random bits; **2^{n^δ} bad strings**

RL

- Recall: probabilistic Turing Machine
  - deterministic TM with extra tape for "coin flips"
- **RL** (Random Logspace)
  - $L \in$ **RL** if there is a probabilistic logspace TM M:
    - $x \in L \Rightarrow Pr_y[M(x,y) \text{ accepts}] \geq ½$
    - $x \notin L \Rightarrow Pr_y[M(x,y) \text{ rejects}] = 1$
  - important detail #1: only allow one-way access to coin-flip tape
  - important detail #2: explicitly require to run in polynomial time

RL

- **L** $\subseteq$ **RL** $\subseteq$ **NL** $\subseteq$ **TIME**($\log^2 n$)
- Theorem (SZ) : **RL** $\subseteq$ **TIME**($\log^{3/2} n$)

- Recall: STCONN is **NL**-complete.
- Undirected STCONN: given an **undirected** graph G = (V, E), nodes s, t, is there a path $s \rightarrow t$

<u>Theorem</u>: USTCONN $\in$ **RL**

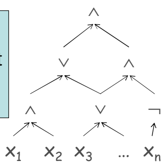
Undirected STCONN

- Proof sketch: (in Papadimitriou)
  - add self-loop to each vertex (technical reasons)
  - start at s, take a random walk for 2|V||E| steps, accept if see t
  - Lemma: expected **return time** for any node i is $2|E|/d_i$
  - suppose $s=v_1, v_2, \ldots, v_n=t$ is a path
  - expected time from v_i to v_{i+1} is $(d_i/2)(2|E|/d_i) = |E|$
  - expected time to reach $v_n \leq |V||E|$
  - Pr[fail reach t in 2|V||E| steps] ≤ ½

A motivating question

- Central problem in logic synthesis:

> • given Boolean circuit C, integer k
>
> • is there a circuit C' of size at most k that computes the same function C does?



- Complexity of this problem?
  - **NP**-hard? in **NP**? in **coNP**? in **PSPACE**?
  - complete for any of these classes?

Outline

- Oracle Turing Machines
- The Polynomial-Time Hierarchy (**PH**)
- Quantified SAT
- Complete problems for classes in **PH**, **PSPACE**

Oracle Turing Machines

- Oracle Turing Machine (OTM):
  - multitape TM M with special "query" tape
  - special states $q_?$, q_{yes}, q_{no}
  - on input x, with oracle language A
  - M^A runs as usual, except…
  - when M^A enters state $q_?$:
    - y = contents of query tape
    - $y \in A \Rightarrow$ transition to q_{yes}
    - $y \notin A \Rightarrow$ transition to q_{no}

Oracle Turing Machines

- Nondeterministic OTM
  - defined in the same way
  - (transition relation, rather than function)
- oracle is like a subroutine, or function in your favorite PL
  - but each call counts as single step

  e.g.: given $\varphi_1, \varphi_2, \ldots, \varphi_n$ are even # satisfiable?
  - poly-time OTM solves with SAT oracle

Oracle Turing Machines

Shorthand #1:

- applying oracles to entire complexity classes:
  - complexity class **C**
  - language A

  $\mathbf{C}^A$ = {L decided by OTM M with oracle A with M "in" **C**}
  - example: $\mathbf{P}^{SAT}$

Oracle Turing Machines

Shorthand #2:

- using complexity classes as oracles:
  - OTM M
  - complexity class **C**
  - $M^{\mathbf{C}}$ decides language L if for some language $A \in \mathbf{C}$, M^A decides L

Both together: $\mathbf{C}^{\mathbf{D}}$ = languages decided by OTM "in" **C** with oracle language from **D**

exercise: show $\mathbf{P}^{SAT} = \mathbf{P}^{\mathbf{NP}}$

The Polynomial-Time Hierarchy

- can define lots of complexity classes using oracles
- the following classes stand out
  - they have natural complete problems
  - they have a natural interpretation in terms of alternating quantifiers
  - they help us state certain consequences and containments (more later)

The Polynomial-Time Hierarchy

$\Sigma_0 = \Pi_0 = P$

$\Delta_1 = P^P$ $\Sigma_1 = NP$ $\Pi_1 = coNP$

$\Delta_2 = P^{NP}$ $\Sigma_2 = NP^{NP}$ $\Pi_2 = coNP^{NP}$

$\Delta_{i+1} = P^{\Sigma_i}$ $\Sigma_{i+i} = NP^{\Sigma_i}$ $\Pi_{i+1} = coNP^{\Sigma_i}$

Polynomial Hierarchy $PH = \cup_i \Sigma_i$

The Polynomial-Time Hierarchy

$\Sigma_0 = \Pi_0 = P$

$\Delta_{i+1} = P^{\Sigma_i}$ $\Sigma_{i+i} = NP^{\Sigma_i}$ $\Pi_{i+1} = coNP^{\Sigma_i}$

- Example:
  - MIN CIRCUIT: given Boolean circuit C, integer k; is there a circuit C' of size at most k that computes the same function C does?
  - MIN CIRCUIT $\in \Sigma_2$