Inapproximability of VCG-Based Auction Mechanisms

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The Model

The VCG Mechanism

An auction requires taking in bids and determining the *allocation* of items to bidders as well as the *payments* made by the bidders for the items they receive. The VCG mechanism is the only known general mechanism for determining payments to align with bidder incentives given an allocation algorithm.

The VCG mechanism requires that the allocation mechanism be *maximal-in-range*.

Maximal-In-Range Mechanisms

Any allocation mechanism has a range R of possible allocations given the bids. A mechanism is maximal-in-range if it always chooses an allocation in R which maximizes the social welfare.

Budget Additive Valuations

The type of valuation we focus on is known as budget additive. A budget additive valuation consists of a value v_i for each item and a maximum total budget b. The total value of set S is

$$\min\left(\sum_{i\in S} v_i, b\right)$$

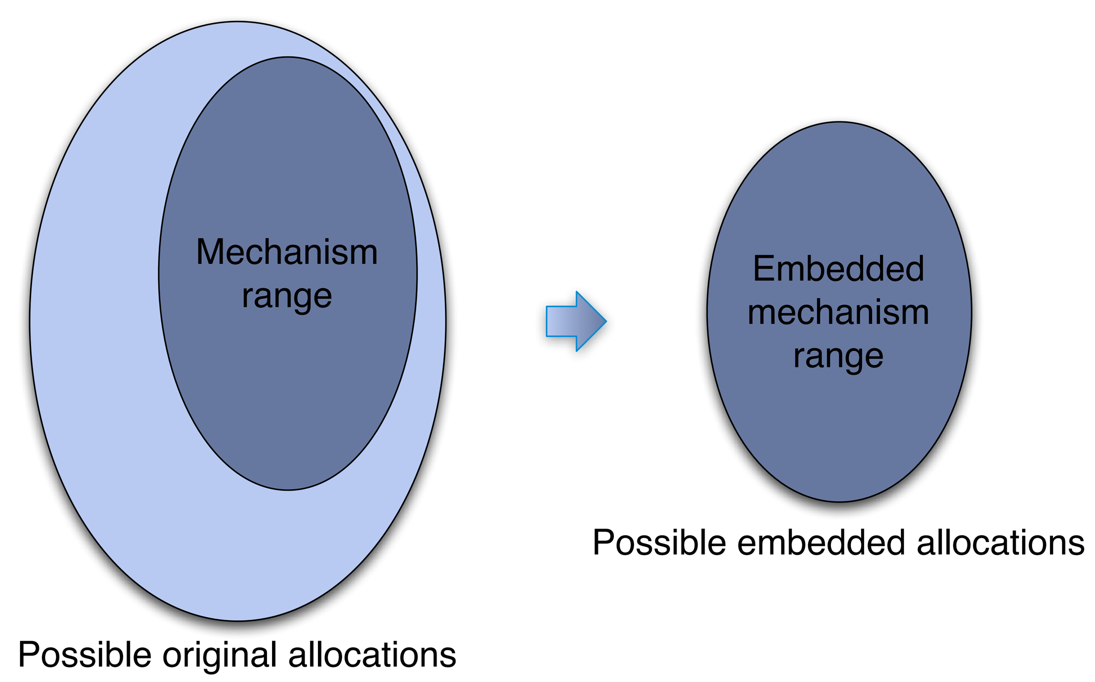
More intuitively, the value of a set of items is the sum of the values of each individual item, up to a maximum upper limit, the budget. This situation may be familiar to frequent diners, as a full sized entree is no longer preferable to a half-sized portion after a large appetizer. In this case, the budget is your appetite.

Techniques

VC Dimension

In previous work by Mossel, Papadimitriou, Schapira and Singer, approximation hardness was shown by demonstrating that any maximal-in-range mechanism with a sufficiently high approximation factor must have a large range.

A large VC dimension allows for a smaller auction to be embedded and solved exactly, an NP-hard feat.



Unfortunately, this approach only shows a bound on the approximation ratio of $2n/(n+1)$ and only for constant number of bidders n. We show a tight bound for all poly-bounded n.

Our Strategy

1. Show a large range by way of a counting argument
2. Use Sauer's lemma indirectly to show a large VC dimension when bidders are combined into a meta-bidder
3. Embed subset sum into the meta-bidder auction

Large Range

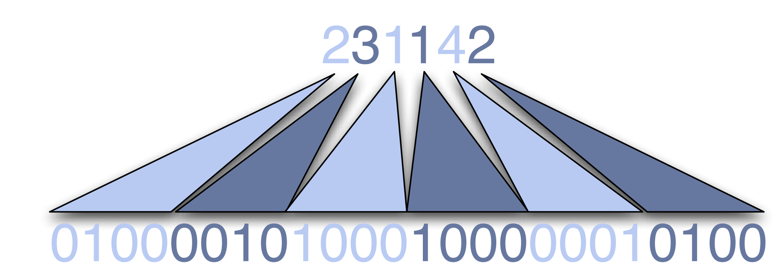
Let T_S be the restriction of the range to items in S. We want to show that $|T_S| > n^{cm}$ for some S and constant c. By comparing how many T_S must contain projections of each range item to how many range items each member of T_S is the projection of, we get the inequality

$$n^{cm}\binom{m}{|S|}n^{m-|S|} < n^m\binom{(1/n+\epsilon)m}{|S|}$$

if $|T_S| < n^{cm}$ and we have an approximation of at least $1/n+\epsilon$. This inequality fails, giving a large range.

Sauer's Lemma

In order to use the standard Sauer's Lemma, we need a 0-1 vector, so we map $n^{|S|} \to 2^{n|S|}$ by expanding each position i into n positions with a single 1 at the ith position.



Sauer's lemma gives us a large VC dimension, and each position determines whether bidder i gets the item, or one of the others does. By focusing on the same relative position, we can find a bidder i^* to single out, leaving the rest as a meta-bidder. The large VC dimension tells us we can solve this meta-bidder auction exactly.

Embedding Subset Sum

Given a subset sum instance $a_1, \ldots, a_\ell$ with target sum k, we give bidder i^* value $2a_j$ for item j and budget $2k$. By giving all other bidders value a_j for item j and unlimited budgets, there is a preference for giving items to bidder i^* until his budget is reached. It is easy to show that the maximum social welfare of

$$\sum_{i=1}^{\ell} a_i + k$$

is achieved iff i^* gets a subset with sum k.

Results

We show that for any class of valuations which includes budget additive valuations, no maximal-in-range polynomial time mechanism can achieve an approximation ratio of

$$\min\left(n, m^{1/2-\epsilon}\right)$$

for any constant ϵ unless NP has polynomial circuits.

A relatively simple maximal-in-range mechanism can be used to achieve an approximation ratio of

$$\min\left(n, 2m^{1/2-\epsilon}\right),$$

so this result is tight.

This result holds for all n bounded by a polynomial in m. For super-polynomial n, either the result holds with weaker assumptions, or m becomes small enough relative to the input size to brute force a solution in poly time.