# The complexity of Boolean formula minimization

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# Formula Minimization Problem

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- So you create a formula $F$ computing $f$
- You want $F$ small

# Formal Definition

**Problem (Minimum Equivalent Expression)**

*Given a formula $F$ and an integer $k$, is there a formula $F'$ equivalent to $F$ of size at most $k$?*

- Size is defined to be the number of occurrences of input variables in the formula.
- In $\Sigma_2^P$

# Example

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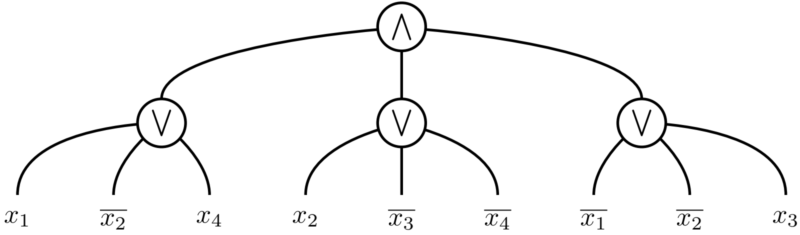

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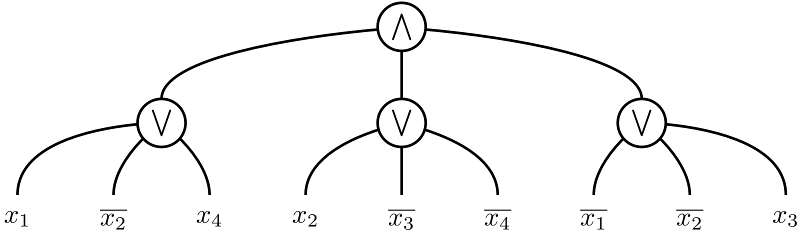


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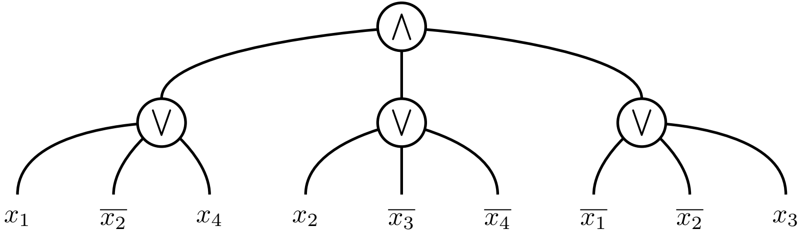


- The size of this formula is $9$
- What's special about $k = 0$ here?

# History of the Problem

- Defined in the early 70's by Meyer and Stockmeyer, inspired the Polynomial Hierarchy
- Clearly coNP-hard
- Proven $\mathrm{P}_{||}^{\mathrm{NP}}$-hard in 1997 (Hemaspaandra and Wechsung)
- DNF version proven $\Sigma_2^{\mathrm{P}}$-complete in 1999 (Umans)
- We show that Minimum Equivalent Expression is $\Sigma_2^P$-complete under Turing reductions, both for unrestricted formulas and for formula restricted to any fixed depth $d \geq 3$

# Why is it hard?

- In the hard direction of the reduction, we need a formula lower bound
- Circuit and formula lower bounds are hard
- We make use of very simple lower bounds

# Outline

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- Can this be done without changing the problem definition?

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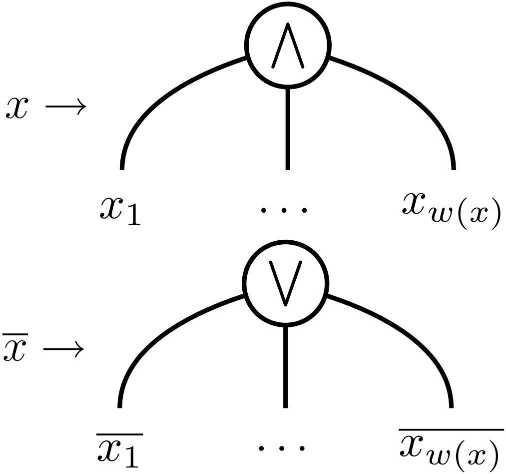
- Can this be done without changing the problem definition?
- Our idea: replace $x$ with $x_1 \wedge \cdots \wedge x_{w(x)}$

# Variable Weighting



# The Results of Weighting

- We start with a formula $F$ computing $f\left(x^{(1)}, \ldots, x^{(n)}\right)$
- We end with a formula $F'$ computing

$$f' = f\left(x_1^{(1)} \wedge \cdots \wedge x_{w\left(x^{(1)}\right)}^{(1)}, \ldots, x_1^{(n)} \wedge \cdots \wedge x_{w\left(x^{(n)}\right)}^{(n)}\right)$$

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**Lemma**

*The minimum formula $F'$ equivalent to $F$ after expanding the weights is at least as large as the minimum weighted formula for $F$.*

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- Each variable $x$ becomes $x_1 \wedge \cdots \wedge x_{w(x)}$ in the expanded form
- Take the $i^*$ such that $x_{i^*}$ occurs least frequently of all $x_i$ in $F'$
- Restrict $x_i = \text{True}$ for $i \neq i^*$ to arrive at $F''$
- Under this restriction, $x_1 \wedge \cdots \wedge x_{w(x)}$ becomes $x_{i^*}$

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- Take the $i^*$ such that $x_{i^*}$ occurs least frequently of all $x_i$ in $F'$
- Restrict $x_i = \text{True}$ for $i \neq i^*$ to arrive at $F''$
- Under this restriction, $x_1 \wedge \cdots \wedge x_{w(x)}$ becomes $x_{i^*}$
- $F''$ is equivalent to $F$
- $F'$ has as many $x_i$ as $w(x)$ times the number of $x_{i^*}$ in $F''$ □

# Modified Succinct Set Cover

## Problem (Modified Succinct Set Cover)

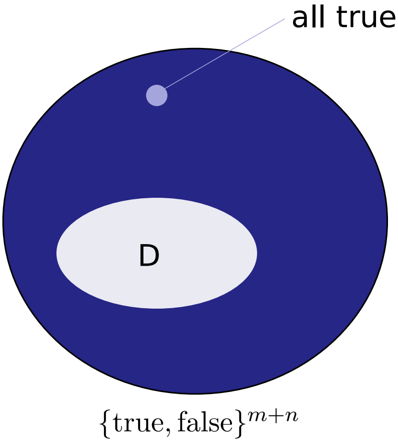
*Given a DNF formula $D$, variables $x_1, \ldots, x_n$ and an integer $k$, where $D$ is a formula on variables $x_1, \ldots, x_n, v_1, \ldots, v_n$, is there a set $I$ of size at most $k$ such that*

$$D \vee \bigvee_{i \in I} \overline{x_i} \equiv D \vee \bigvee_{i=1}^{n} \overline{x_i} \equiv \bigvee_{i=1}^{m} \overline{v_i} \vee \bigvee_{i=1}^{n} \overline{x_i}?$$

- Basically, we want to know how many $\overline{x_i}$ are necessary to cover the assignments not accepted by $D$, other than the all true assignment
- Slight modification of problem used to prove DNF version $\Sigma_2^P$-complete (Umans 1999)

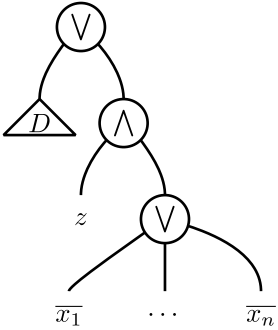
# Succinct Set Cover Visualized

- Each set $\overline{x_i}$ covers half of all points
- None cover the all true point
- How many are necessary to cover the dark blue region?



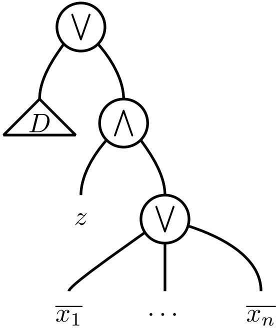
# Overview

- We start with Modified Succinct Set Cover instance $\langle D, x_1, \ldots, x_n, k \rangle$
- We create the Minimum Equivalent Expression instance with formula $D \vee (z \wedge \bigvee_i \overline{x_i})$ and size target $|\widehat{D}|_w + w(z) + k$
- Finding $|\widehat{D}|_w$ necessitates a Turing reduction



# Overview

- When $z$ is false, the formula becomes simply $D$
- When $z$ is true, it "unlocks" a portion computing the set cover



# The Easy Direction

**The question asked by reduction**

Is there a formula for $D \vee (z \wedge \bigvee_i \overline{x_i})$ of size $|\widehat{D}|_w + w(z) + k$?

Easy direction: The Modified Succinct Set Cover is positive, so

$$D \vee \bigvee_{i \in I} \overline{x_i} \equiv D \vee \bigvee_{i=1}^{n} \overline{x_i}$$

which gives us the formula

$$\widehat{D} \vee \left( z \wedge \bigvee_{i \in I} \overline{x_i} \right) \equiv D \vee \left( z \wedge \bigvee_{i=1}^{n} \overline{x_i} \right)$$

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- $z$ is weighted such that $2w(z) > |\widehat{D}|_w + w(z) + k$

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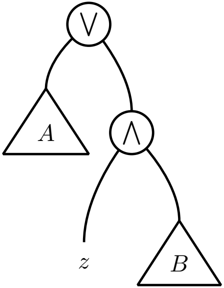
- $z$ is weighted such that $2w(z) > |\widehat{D}|_w + w(z) + k$
- The position of the $z$ is proven through case analysis and requires slight modifications to the reduction

# Consequences of $z$ Position

- Given this positioning, $A \equiv D$ and $B$ computes the set cover
- If we weight all variables other than the $x_i$ by more than $k$, $B$ can only contain $x_i$ variables



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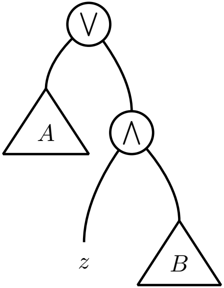



**Lemma**

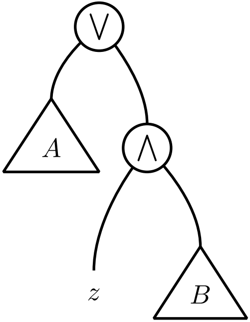
*A minimum formula accepting a set $S$ but not the all true assignment is of the form $\bigvee_i \overline{x_i}$*

# Wrapping it up

**The question asked by reduction**

Is there a formula for $D \vee (z \wedge \bigvee_i \overline{x_i})$ of size $|\widehat{D}|_w + w(z) + k$?

- $A \equiv D$, so $|A|_w \geq |\widehat{D}|_w$
- So size is at least $|\widehat{D}|_w + w(z) + |B|_w$
- As shown above, $|B|_w \leq k$ only if the Modified Succinct Set Cover instance is positive



# Open Problems

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A full version of this paper is available at
`http://www.cs.caltech.edu/~dave/papers/`